The residue matrix of the pole at $E = M_S$ is

$$R_S = \frac{1}{2}\Gamma_S P_S + \frac{i}{4}\frac{\Gamma_S\Gamma_L(P_L P_S + P_S P_L)}{M_L - M_S}; \qquad (5)$$

and similarly at $E = M_L$:

$$R_L = \frac{1}{2}\Gamma_L P_L + \frac{i}{4}\frac{\Gamma_S\Gamma_L(P_L P_S + P_S P_L)}{M_S - M_L}. \qquad (6)$$

The transition amplitude of Φ_L to a state $|n\rangle$ is

$$\langle n|H|\Phi_L\rangle \equiv \langle n|(2R_L/\Gamma_L^{1/2})|\Phi_L\rangle,$$

and of Φ_S to a state $|n\rangle$ is

$$\langle n|H|\Phi_S\rangle \equiv \langle n|(2R_S/\Gamma_S^{1/2})|\Phi_S\rangle.$$

From these relations it is easy to derive the following sum rule:

$$(\Gamma_S\Gamma_L)^{1/2}\chi\left[\frac{\delta m}{\delta m + \frac{1}{2}i\Gamma_S}\right] = \sum_n \langle n|H|\Phi_S\rangle^* \langle n|H|\Phi_L\rangle, \qquad (7)$$

where $\delta m = m_L - m_S$. In (7) only terms linear in χ are retained and Γ_L was ignored compared with Γ_S.

This sum rule differs considerably from (1). It states, for instance, that the overlap can approach unity if $\delta m \gg \Gamma_S$. This statement still obtains if one keeps all orders of χ in (7). Thus there appears to be no restriction once the mass difference is large compared with the widths.

There is a further restriction on P_S and P_L, that is on $|\Phi_S\rangle$ and $|\Phi_L\rangle$, if one wants to interpret K_L and K_S resonant states as linear combinations of K_1 and K_2, the states of definite CP. Then $|\Phi_S\rangle$ and $|\Phi_L\rangle$ must be linear combinations of two states $|\Phi_1\rangle$ and $|\Phi_2\rangle$, where $|\Phi_1\rangle$ and $|\Phi_2\rangle$ have definite CP properties. This further assumption does not change the sum-rule result.

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VENEZIANO MODEL FOR π-ρ SCATTERING*

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A Veneziano model for the odd-normality (π) contributions to $\pi\rho$ double-charge-exchange scattering is found. In the model, the leading trajectory is not parity doubled and its residues satisfy the factorization condition. An argument is given for the existence of high-mass exotic resonances.

We have investigated some of the problems which occur when one attempts to construct a model like Veneziano's[1] for scattering amplitudes in which more than one of the particles has spin. The new features are the possibility of parity doubling and the question of factorization of the residues (nondegeneracy) along the Regge trajectory. As an example, we have studied $\pi\rho$ double-charge-exchange scattering, $\pi^+ + \rho^- \to \pi^- + \rho^+$. Resonances occur only in the s and u channels, which are identical, and we are freed from the complications of dealing with channels which communicate with the ρ or the vacuum.

Let the pion momenta be p and p', the ρ momenta be q and q', and the final polarization vectors be ϵ and ϵ', with $\epsilon\cdot q = \epsilon'\cdot q' = 0$. The primes refer to the final state. The scattering amplitude is $\epsilon'^\nu\epsilon^\mu T_{\mu\nu}$, and $T_{\mu\nu}$ can be expanded in terms of four scalar invariant amplitudes:

$$T_{\mu\nu} = A(s,u)P_\mu P_\nu + B(s,u)[P_\mu q_\nu + q_\mu' P_\nu] + C(s,u)q_\nu q_\mu' + D(s,u)g_{\mu\nu}, \qquad (1)$$

where $P = \frac{1}{2}(p + p')$. Under exchange of s and u, A, C, and D are even and B is odd.

It is convenient to use the familiar "parity-conserving" helicity amplitudes as defined by Gell-Mann et al.[2,3]:

$$m^2T_{00}=AQ^2s+(2B-A)EWQ^2(1-z)+[C-B+\tfrac{1}{4}A]E^2Q^2(1-z)^2+D(Q^2-E^2z),$$
$$m\sqrt{2}T_{01}^{-}=DE+[C-B+\tfrac{1}{4}A]EQ^2(1-z)+\tfrac{1}{2}(2B-A)Q^2W,$$
$$T_{11}^{-}=[C-B+\tfrac{1}{4}A]Q^2z-D,$$
$$T_{11}^{+}=-[C-B+\tfrac{1}{4}A]Q^2. \qquad (2)$$

Consider, for simplicity, a model in which only the pion (A_1) trajectory contributes. A realistic model, of course, should include the ω trajectory also. We use the parameters suggested by Ademollo, Veneziano, and Weinberg,[4] namely, $m_\pi=0$ and $\alpha_\pi(s)=s$ in units such that $2m_\rho^2=1$. Then the invariant amplitudes will be written as a linear combination of terms of the type

$$F_m^{kl}(s,u)=\Gamma(k-s)\Gamma(l-u)/\Gamma(m-s-u). \qquad (3)$$

For example,

$$A(s,u)=\sum_{k,l,m}A_m^{kl}F_m^{kl}(s,u), \qquad (4)$$

and similarly for B, C, and D.

The sum in (4) must be restricted by the condition $m\leqslant k+l$, so that the residues of the poles in s are finite-order polynomials in z, and obey $m\geqslant k$, and $m\geqslant l$ so that for fixed u, A, B, C, and D grow no faster than $s^{\alpha_\pi(u)}$. Furthermore, it follows from (2) that the pion pole cannot appear in D, so that $D^{0l}=D^{k0}=0$.

We choose as our representation[5]

$$A=A_0^{00}F_0^{00}+A_1^{10}(F_1^{10}+F_1^{01})+A_2^{20}(F_2^{20}+F_2^{02})+A_1^{11}F_1^{11}+A_2^{11}F_2^{11},$$

$$B=B_1^{10}(F_1^{10}-F_1^{01})+B_2^{20}(F_2^{20}-F_2^{02}),$$

$$C=C_0^{00}F_0^{00}+C_1^{10}(F_1^{10}+F_1^{01})+C_2^{20}(F_2^{20}+F_2^{02})+C_1^{11}F_1^{11}+C_2^{11}F_2^{11},$$

$$D=D_1^{11}F_1^{11}+D_2^{11}F_2^{11}. \qquad (5)$$

In order that the physical helicity amplitudes be bounded, for large s at fixed u, by $s^{\alpha_\pi(u)}$ it is necessary that the amplitudes defined in (2) satisfy[6]

$$T_{11}^{-}+T_{11}^{+}\sim s^{\alpha_\pi(u)+1},$$
$$T_{11}^{-}-T_{11}^{+}\sim s^{\alpha_\pi(u)},$$
$$T_{01}^{-}\sim s^{\alpha_\pi(u)+\frac{1}{2}},$$
$$T_{00}\sim s^{\alpha_\pi(u)}. \qquad (6)$$

Expand A, B, C, and D at large s as

$$A=a_1(u)s^{\alpha_\pi(u)}+a_2(u)s^{\alpha_\pi(u)-1}+\cdots,$$

etc. Then the conditions (6) are satisfied by

$$a_1(u)=4c_1(u),$$
$$b_1(u)=2c_1(u),$$
$$a_2(u)+4b_2(u)+4c_2(u)+8d_1(u)=0. \qquad (7)$$

Conditions (7) must hold at all u in order that the spin of the leading exchanged trajectory be in fact no greater than $\alpha_\pi(u)$. Because of the form of the representation (5), the functions in (7) are finite-order polynomials in u, and therefore[7] can be satisfied by a finite number of conditions on A_m^{kl}, B_m^{kl}, C_m^{kl}, and D_m^{kl}. In fact, the rules (7) hold provided

$$A_1^{10}+A_2^{20}+A_0^{00}=4[C_1^{10}+C_2^{20}+C_0^{00}],$$
$$B_1^{10}+B_2^{20}+-2[C_1^{10}+C_2^{20}+C_0^{00}],$$
$$8D_1^{11}=-16[C_1^{10}+C_2^{20}+C_0^{00}]+[2A_2^{20}-A_2^{11}]+4[2C_2^{20}-C_2^{11}],$$
$$A_1^{11}=0,$$
$$C_1^{11}=0. \qquad (8)$$

Next we may impose the condition that the leading trajectory not be parity doubled, i.e., that there is no 1^- particle at $s=1$, etc. We cannot require that all the daughter trajectories have only particles of a single normality without adding an infinite number of terms in (4), as has been shown by Freund and Schonberg in a related problem.[7] The condition can be obtained from (2) and the partial-wave expansions of T_{00}, T_{01}^{-}, and $T_{11}^{\pm}$:

$$T_{00}=\sum_{j=0}^{\infty}(2j+1)P_j(z)t_{00}^{j-}(s),$$
$$T_{01}^{-}=\frac{1}{\sqrt{2}}\sum_{j=1}^{\infty}\frac{(2j+1)}{[j(j+1)]^{1/2}}P_j^{1}(z)t_{01}^{-}(s),$$
$$T_{11}^{\pm}=\sum_{j=1}^{\infty}\frac{(2j+1)}{j(j+1)}\{[P_j^{1}(z)+zP_j^{11}(z)]t_{11}^{j\pm}(s)-P_j^{11}(z)t_{11}^{j\mp}(s)\}. \qquad (9)$$

Here $t_{\mu\nu}{}^{j\pm}$ are elements of the partial-wave matrix between states the magnitude of whose helicities are μ and ν, spin j, and parity $\pm(-1)^j$. The $t_{\mu\nu}{}^{\pm}$ are all normalized identically. Then, if the pole at $s=n$ does not appear in $t_{11}{}^{n+}$, the residue in $D(s,u)$ grows as u^{n-1} instead of u^n for large u. Therefore, the no-parity-doubling condition is

$$D_1{}^{11}=0. \tag{10}$$

A more complicated question is the factorization of the leading trajectory. We require that the residues of the poles of the matrix $t^{j-}(s)$ factor at $s=j$. From (9), it is seen that this is equivalent to requiring that the coefficient of z^n in the residue of the pole at $s=n$, multiplied by the corresponding coefficient of z^{n-1} in $T_{11}{}^-$, be equal to the square of the coefficient of z^{n-1} in the residue at $s=n$ of $\sqrt{2}T_{01}{}^-$. We obtain these coefficients from our representation (5). They turn out to depend on only three combinations of the $A_m{}^{kl}$, etc., namely

$$D=D_2{}^{11},$$
$$C=C_1{}^{10}+C_2{}^{20}+C_0{}^{00},$$
$$4M=(A_2{}^{11}-2A_2{}^{20}). \tag{11}$$

Factorization is achieved provided

$$X_{00}X_{11}=2(X_{01})^2, \tag{12}$$

where

$$X_{00}=(n+\tfrac{1}{2})(n-\tfrac{1}{2})^2[2C+M]-\tfrac{1}{2}(n+\tfrac{1}{2})^2D -(n+\tfrac{1}{2})^2(n+1)C-(n-\tfrac{1}{2})^4C/n,$$
$$X_{11}=-n[D+2(n-1)C],$$
$$X_{01}=\tfrac{1}{2}n^{1/2}[(n+\tfrac{1}{2})D+2(n+\tfrac{1}{2})(n-1)C -(n-\tfrac{1}{2})^2(2C+M)]. \tag{13}$$

Equation (12) must hold for all n. In general, one might not expect a solution, since (12) is an infinite number of conditions on a small number of coefficients $A_m{}^{kl}$, etc. However, our model, with its limited number of terms, admits two solutions:

$$C=M=0,$$
$$D\neq 0, \tag{14}$$

and

$$D=2C,$$
$$M=-4C. \tag{15}$$

Thus, it is possible to achieve factorization and no parity doubling with a finite number of terms. Notice that we cannot require that the daughters factor, or even, unlike Freund and Schonberg, that they are not parity doubled.[8]

Next, let us attempt to impose the Adler self-consistency condition. There are no threshold poles in our problem, so the physical helicity amplitudes should vanish when $s=u=\frac{1}{2}$. In our representation, this reduces to $D=0$ at that point. There are no conditions on A, B, and C. Lovelace[9] and Ademollo, Veneziano, and Weinberg[4] have suggested that each invariant amplitude contain at least one term which vanishes automatically at this point, as do those of our terms with $m=0$ and $m=1$. [However, we cannot have terms of this type only in A, B, and D since the conditions (7) and (10) imply immediately that $B(s,u)=0$, and the leading trajectory is not there at all.] The only term in $D(s,u)$ is $D_2{}^{11}$, and therefore $D(\frac{1}{2},\frac{1}{2})=0$ requires $D_2{}^{11}=0$. This does not violate the condition[10] that the leading trajectory have only one parity, but is clearly inconsistent with either of our solutions (14) and (15) to the factorization problem unless, again, everything is zero.

Note, however, that one can add to D a very low satellite term, $D_m{}^{kl}F_m{}^{kl}(s,u)$, where $m<l+2$ in order to achieve $D(\frac{1}{2},\frac{1}{2})=0$, without affecting the parameters of the leading trajectory, thereby retaining factorization. Essentially this is because the Adler condition is a condition on the residues of all the s-wave particles in the model, most of which lie on daughter trajectories, and is therefore <u>independent</u> of the problem of describing correctly the leading trajectory. Thus it is no longer clear that the assumption is compelling that the self-consistency condition is fulfilled by the blowing up of the gamma functions in the denominators, as suggested in Ref. 4, rather than by cancellation. Nevertheless, the quantization conditions derived there seem to agree strikingly with current algebra and with experiment.

Finally, there remains a more general feature which should be observed. Because $u=-2Q^2z+\text{const}$ and because the leading term in the residue of the pole at $s=n$ is u^n, the signs of the residues of all particles along the leading trajectory alternate. This is a general feature of all processes without t-channel resonances, i.e., those in which the t channel has exotic quantum numbers.

This alternation is a success of the double charge exchange part of the π-π model of Lovelace,[9] since the s-channel isospin resolution of this amplitude is

$$\tfrac{1}{2}T^{(0)}-\tfrac{1}{3}T^{(1)}+\tfrac{1}{6}T^{(2)}.$$

Therefore, the alternating signs along his degenerate ρ-f_0 trajectory simply mean that $(-1)^I$ alternates, as it does. The same is true in other cases.[10] Here, however, we want the π and the A_1 to have the same isospin. Addition of further satellite terms $F^{kl}{}_m(s,u)$ cannot change this feature, and the only recourse is to include an $I=2$ t-channel trajectory. If the trajectory is low enough, the mass of the lowest $I=2$ particle will be high enough not to disagree with present experiments. We have verified that with the inclusion of such a low-lying trajectory there exists a solution, similar in form to the above, with factorization and without parity doubling for the leading trajectory, satisfying the Adler condition, and with the same sign for the residues of all the particles on the leading trajectory. A detailed and more general account will be published elsewhere.

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[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]M. Gell-Mann, M. L. Goldberger, F. E. Low, E. Marx, and F. Zachariasen, Phys. Rev. 133, B145 (1964), Appendix A.

[3]In our notation, $s=(p+q)^2$, $u=(p-q')^2$, $W=\sqrt{s}$, and E, Q, m, and z are the energy, momentum, mass, and cosine of the scattering angle of the ρ in the center-of-mass frame.

[4]M. Ademollo, S. Weinberg, and G. Veneziano, Phys. Rev. Letters 22, 83 (1969).

[5]Terms in $F_2{}^{21}$ could also be included, but we have verified that they do not change the results of this paper. A smaller number of terms makes it impossible to impose the conditions below unless the pion residue vanished.

[6]The bounds on the helicity amplitudes could be insured by having $A,B,C,D\sim s^{\alpha-2}$ but then the pion pole would not appear, i.e., the pion would be "nonsense choosing."

[7]P. G. O. Freund and R. Schonberg, University of Chicago Report No. EFI 69-1, 1969 (to be published).

[8]If the daughters generally factored in models of this type, then the amplitudes for equal-mass scattering at $s=0$ would be simply related to Gegenbauer polynomials, as shown by J. Bronzan and C. E. Jones, Phys. Rev. Letters 21, 564 (1968); whereas functions of the type $F_m{}^{kl}(s,u)$ are clearly not Gegenbauer polynomials at $s=0$. We thank C. Edward Jones and Gabriele Veneziano for discussing this point with us.

[9]C. Lovelace, CERN Report No. TH950, 1968 (to be published).

[10]For example, $\pi\Sigma$ or $K\bar{K}$ elastic scattering. See H. Lipkin, to be published.