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RENORMALIZATION OF THE EFFECTIVE *g* FACTOR OF MAGNONS*

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Within the framework of the Heisenberg model we have calculated the renormalization of the effective g factor of magnons in cubic ferromagnets and antiferromagnets. We find a $f(k)T^{3/2}$ enhancement of g in ferromagnets, where $f(k)\sim k^2$ for small k, and a $-\frac{1}{2}\chi_\parallel/\chi_\perp$ depression of g for all antiferromagnetic magnons. We estimate that this effect will have experimental consequences of 10% or more at reasonable temperatures.

The temperature-dependent renormalization of magnon energies in the absence of an externally applied magnetic field is a well-known phenomenon, both from a theoretical and an experimental viewpoint. In the case of the Heisenberg ferromagnet, Bloch[1] and Keffer and Loudon[2] have developed a theory of spin-wave renormalization. Such effects have been observed, for example, by Weber and Tannenwald[3] in Permalloy films. In the case of the Heisenberg antiferromagnet, White[4] has developed a theory to explain the temperature-dependent shift of the magnon-exciton sideband in MnF_2 observed by Greene et al.[5] In this Letter, we consider the renormalization of magnon energy in the presence of an externally applied magnetic field. We show that to lowest order in the applied field this results in a renormalization of the effective g factor of the magnon.

We apply the Holstein-Primakoff theory of the Heisenberg ferromagnet and antiferromagnet, including terms fourth order in the boson operators. Generalizing the work of Ref. 1 to include the effects of a magnetic field applied along the axis of magnetization, we find the magnon energy in a cubic ferromagnet with nearest-neighbor interactions to be given by

$$\epsilon_{\vec{k}}=\omega_{\vec{k}}+\mu_0 H+\frac{2Jz}{N}\sum_{\vec{q}}\{\gamma(\vec{k})+\gamma(\vec{q})-\gamma(\vec{0})-\gamma(\vec{k}-\vec{q})\}n_{\vec{q}} \quad (1)$$

In this expression $\omega_{\vec{k}}=2JzS[1-\gamma(\vec{k})]$, J is the nearest-neighbor exchange constant, z is the number of nearest neighbors, S is the spin of the magnetic ions, and $z\gamma(\vec{k})=\sum_{\vec{\delta}}e^{i\vec{k}\cdot\vec{\delta}}$. $\vec{\delta}$ is a vector from a given site to a nearest-neighbor site. $n_{\vec{q}}$, the boson occupation number, equals $[\exp(\beta\epsilon_{\vec{q}})-1]^{-1}$, where $\beta=(k_BT)^{-1}$, and $\mu_0=g_0\mu_B$. Normally $g_0\approx 2$.

Equation (1) holds at temperatures below about one-half the Curie temperature and for temperatures with thermal energy larger than any anisotropy energies, so that anisotropy effects may be safely neglected.

To obtain the g-factor renormalization we make a Taylor expansion of the occupation number about $H=0$. The term linear in H gives the correction Δg to the effective g factor of the magnon.[6] Writing

$$\epsilon_{\vec{k}}=\epsilon_{\vec{k}}^{(0)}(T)+(g_0+\Delta g_{\vec{k}})\mu_BH+O(H^2),$$

we have

$$\frac{\Delta g_{\vec{k}}}{g_0}\approx-\frac{2Jz}{k_BT}\frac{1}{N}\sum_{\vec{q}}\{\gamma(\vec{k})+\gamma(\vec{q})-\gamma(\vec{0})-\gamma(\vec{k}-\vec{q})\}\times n_{\vec{q}}(1+n_{\vec{q}}),\qquad(2)$$

where $n_{\vec{q}}$ is evaluated for zero field. We note that the $\vec{k}=\vec{0}$ mode is unaffected by renormalization effects. This is as should be, since on general principles the uniform precession frequency for an isotropic ferromagnet should be independent of any internal interactions.

We may evaluate Eq. (2) for temperatures such that $\omega_{\vec{q}}\approx Dq^2$ for thermal magnons. Here $D=2JSa^2$, where a is the lattice constant. Within this approximation we may further write

$$\frac{\Delta g_{\vec{k}}}{g_0}\approx\frac{2Jz[1-\gamma(\vec{k})]}{k_BT}\frac{1}{N}\sum_{\vec{q}}[1-\gamma(\vec{q})]n_{\vec{q}}(1+n_{\vec{q}})$$
$$=\frac{[1-\gamma(\vec{k})]}{Sk_BT}\frac{1}{N}\sum_{\vec{q}}\omega_{\vec{q}}n_{\vec{q}}(1+n_{\vec{q}})$$
$$\approx\frac{3[1-\gamma(\vec{k})]}{16S}\lambda_c\zeta\left(\frac{3}{2}\right)\left(\frac{k_BT}{2\pi SJ}\right)^{3/2},\qquad(3)$$

where $\lambda_c=1,\frac{1}{2},\frac{1}{4}$ for sc, bcc, and fcc lattices, respectively. In deriving Eq. (3) we have taken the limit

$$\frac{1}{N}\sum_{\vec{q}}\to\frac{\lambda_c a^3}{(2\pi)^3}\int_{BZ}d\vec{q},$$

and have extended the integral over the Brillouin zone out to infinity.

We may rewrite Eq. (3) using a well-known result from molecular field theory[7]:

$$J=3k_BT_c/2zS(S+1).\qquad(4)$$

Substituting this into Eq. (3), we obtain

$$\frac{\Delta g_k}{g_0}=\frac{3[1-\gamma(\vec{k})]\lambda_c\zeta(\frac{3}{2})}{16S}\left[\frac{z(S+1)T}{3\pi T_c}\right]^{3/2}.\qquad(5)$$

EuO and EuS are Heisenberg ferromagnets to which this theory may be applied. They both have $S=\frac{7}{2}$ and the magnetic ions alone have a fcc structure, so $z=12$ and $\lambda_c=\frac{1}{4}$. Further,

$$\gamma(\vec{k})=\tfrac{1}{3}[\cos(\tfrac{1}{2}k_xa)\cos(\tfrac{1}{2}k_ya)+\cos(\tfrac{1}{2}k_ya)\times\cos(\tfrac{1}{2}k_za)+\cos(\tfrac{1}{2}k_za)\cos(\tfrac{1}{2}k_xa)].$$

When $\gamma(\vec{k})=0$, which occurs along certain symmetry lines of the Brillouin zone, $\Delta g_k/g_0$ is maximized. In that case,

$$\left(\frac{\Delta g_{\vec{k}}}{g_0}\right)_{\max}\approx0.48\left[\frac{T}{T_c}\right]^{3/2}.$$

Thus it should be feasible to measure magnon g-factor renormalization in EuO and EuS by direct observation (i.e., neutron scattering of magnons), since this is a 17% effect for $T=\frac{1}{2}T_c$.

We also note that the effect of g-factor renormalization may be observed by spin wave resonance in thin films.[8] The resonance condition is $\hbar\omega=\epsilon_{\vec{k}}$, or

$$\hbar\omega=(g_0+\Delta g_{\vec{k}})\mu_BH+\epsilon_{\vec{k}}^{(0)}.$$

Then, since for small k we have $\epsilon_{\vec{k}}^{(0)}=Ak^2-Bk^4$ and $\Delta g_{\vec{k}}/g_0=Ck^2$, we have

$$g_0\mu_BH\approx\hbar\omega-(A+\hbar\omega C)k^2+(B+AC)k^4.$$

Evaluating the relevant parameters for Permalloy, as in Ref. 3, we have $\hbar\omega C\ll A$, but AC is 17% of B at room temperature. Consequently we predict an appreciable $T^{3/2}$ dependence in the coefficient of the k^4 term of spin waves observed by thin-film resonance in Permalloy.

We now turn to the Heisenberg antiferromagnet. We may take over the results of Oguchi[9] and White,[4] slightly generalized to include the presence of an applied magnetic field. Neglecting anisotropy and considering only nearest-neighbor interactions, we have

$$\epsilon_{\vec{k}}^{\alpha,\beta}=\omega_{\vec{k}}\left[1-\frac{1}{NS}\sum_{\vec{q}}\frac{\omega_{\vec{q}}}{2JzS}(n_{\vec{q}}^{\alpha}+n_{\vec{q}}^{\beta})\right]\pm\left[g_0\mu_BH+\frac{2Jz}{N}\sum_{\vec{q}}(n_{\vec{q}}^{\alpha}-n_{\vec{q}}^{\beta})\right]. \qquad (6)$$

Here $\omega_{\vec{q}}=2JzS(1-\gamma_{\vec{q}}^2)^{1/2}$, and N is the total number of atoms in the crystal. The other quantities are defined in analogous fashion to the ferromagnetic case. The α magnons have their energies raised in the presence of a magnetic field along the z axis (which is the axis of magnetization); the β magnons have their energies lowered. The quantities $n_{\vec{q}}^{\alpha}$ and $n_{\vec{q}}^{\beta}$ are the thermal occupation numbers for α and β magnons of wave vector $\vec{q}$.

The correction to the g factor arises solely from the $n_{\vec{q}}^{\alpha}-n_{\vec{q}}^{\beta}$ term. Explicitly,

$$\frac{\Delta g_{\vec{k}}}{g_0}=-\frac{4Jz}{k_BT}\frac{1}{N}\sum_{\vec{q}}n_{\vec{q}}(1+n_{\vec{q}}), \qquad (7)$$

where $n_{\vec{q}}$ is evaluated at zero field. We note that this expression is correct even with anisotropy present, in which case $\omega_{\vec{q}}=[(2JzS+g_0\mu_BH_A)^2-(2JzS\gamma_{\vec{q}})^2]^{1/2}$, where H_A is the anisotropy field. In contrast to the ferromagnetic case, the renormalized g factor is constant throughout the Brillouin zone.

One may also obtain Eq. (7) by applying the random-phase approximation (RPA) to the equations of motion for the spin operators. This is equivalent to replacing quantities such as $\langle S_{lz}S_{m+}\rangle$ by $\langle S_{lz}\rangle\langle S_{m+}\rangle$, where l and m refer to different spin sites, and then calculating the field dependence of $2Jz\langle S_{lz}\rangle$ by expanding $\langle S_{lz}\rangle$ using the spin-wave theory. We note that the exchange energy is the origin of both the RPA and the spin-wave g-factor renormalization. For $\vec{k}=\vec{0}$ the RPA is also equivalent to the work of Keffer and Kittel[10] on the uniform antiferromagnetic resonance mode. Thus we may rewrite Eq. (7) in terms of the expression given by Keffer and Kittel. We then have

$$\Delta g_{\vec{k}}/g_0=-\tfrac{1}{2}\chi_{\parallel}\chi_{\perp}, \qquad (8)$$

where $\chi_{\parallel}$ and $\chi_{\perp}$ are the parallel and perpendicular susceptibilities, respectively, of the antiferromagnet in the normal state. They may often be obtained from tables. In the case of MnF_2 we may obtain these quantities from Foner.[11] For $T=30°K$, we find $\Delta g_{\vec{k}}/g_0\approx-0.10$, which is an appreciable amount.

We remark that it seems to be a coincidence that the RPA gives the same result as the more rigorous spin-wave renormalization theory. It is well-known, for example, that the RPA applied to the renormalization of magnon energies in ferromagnets gives a $T^{3/2}$ renormalization,[12] whereas a $T^{5/2}$ renormalization is predicted by the spin-wave renormalization theory[13] and is observed experimentally.[3]

Although g-factor renormalization for $\vec{k}=\vec{0}$ magnons in the antiferromagnet is an already observed phenomenon,[14,15] it would be desirable to establish the effect for $\vec{k}\neq\vec{0}$ magnons. As in the case of the ferromagnet, neutron scattering is applicable. In addition, since the renormalized g factor is constant throughout the Brillouin zone, exciton-magnon sidebands such as reported in Ref. 5 will shift uniformly on application of an external magnetic field; there will be no change in line shape. This enables one to measure the effects of temperature upon the magnon g factor, since we do not expect the exciton g factor to show any significant temperature dependence.

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LOW-TEMPERATURE THERMAL CONDUCTIVITY OF POLYETHYLENE

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The thermal conductivity of a sample of polyethylene has been measured in the temperature range below 1°K. The results obtained are inconsistent with previous theories offered to explain thermal conduction in these materials at higher temperatures; a simple modification to the theory is suggested which is consistent with the current data.

The thermal conductivity of polyethylene has been measured in the temperature range 1-4.5°K by several investigators.[1,2] Their data indicate that the thermal conductivity in this temperature range can be fitted by an expression used by Chang and Jones in their analysis of the low-temperature thermal conductivity of glasses.[3] Thermal conductivity measurements have been made on a sample of polyethylene in the range 0.5-1.4 °K and our results below 1°K deviate from the Chang and Jones expression, having a lower power dependence on temperature than would be expected from this expression.

The measurements were made on a commercial sample of polyethylene of density 0.947 g/cm^3 or fractional crystallinity about 65%. A standard He^3 cryostat and sample arrangement were used in making the measurements. The sample was in the form of a cylinder of length 3 cm and diameter 1 cm. The top of the sample was attached to a threaded copper post on the bottom of the He^3 chamber, and a heater was wound directly onto the bottom of the sample. Allen-Bradley $\frac{1}{8}$-W resistors of nominal resistance 10 Ω were used as secondary thermometers and were inserted into small holes drilled through the sample at spacings of about 0.8 and 0.5 cm, respectively, for two different experimental runs. These thermometers were calibrated directly against the vapor pressure of liquid He^3. Figure 1 shows the results of measurements made in the two different runs with the above sample. Because of the consistency of the results between the two runs, it is assumed that the relatively small thermometer spacing had little effect on the observed temperature gradients in the sample.

The Chang and Jones expression for thermal conductivity is obtained by assuming, as suggested by Klemens,[4] that only longitudinal phonons contribute to the low-temperature thermal conductivity, and that the mean free path due to structure scattering is $(A/a)k^{-2}$, where k is the

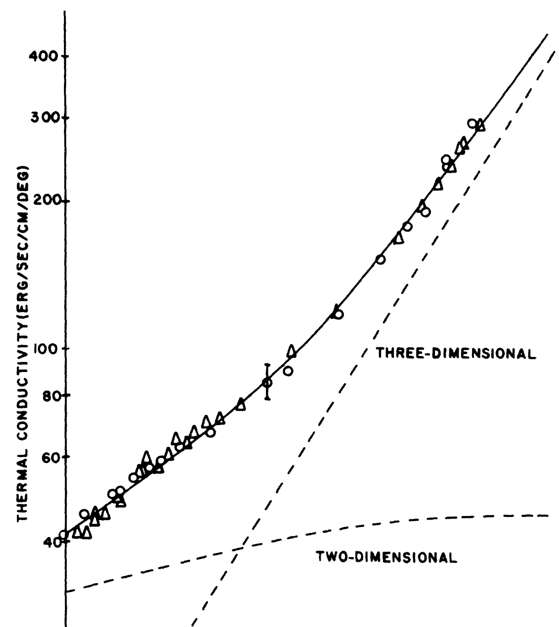

FIG. 1. Thermal conductivity of 65% crystalline polyethylene versus temperature. The open circles and triangles represent the results of two runs on the same sample. The solid curve represents the fit of Eq. (2) to the data using parameters given in the text and the dashed lines represent the separate contributions to κ as given by Eq. (2).