The 4-MeV ring is expected to be achieved in the next few months following the recent completion of the improvements to the Astron accelerator. The new parameters of the Astron accelerator are $E=4$ MeV and $I=800$ A. The 1000-MeV ring is not expected to be achieved for several more years, however, because it requires a new accelerator, probably a betatron using as the injector the Astron accelerator, to accelerate 1000 A to 1000 MeV. In concluding I would like to say that another advantage of the static field compressor is that it can produce a large number of rings per second at a rate depending only on the injection rate from the electron accelerator.

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*Work done under the auspices of the U. S. Atomic Energy Commission.

[1]V. I. Veksler et al., in Proceedings of the Sixth International Conference on High Energy Accelerators, Cambridge, Massachusetts, 14 September 1967 (to be published).

[2]Informal meeting on Electron Ring Accelerators held in Novosibirsk, U. S. S. R., on 5 August 1968.

[3]Since this paper was prepared the following new developments have been reported: Electron rings have been successfully trapped and compressed at Lawrence Radiation Laboratory, Livermore [D. Keefe et al., Phys. Rev. Letters 22, 588 (1969)]. At the 1969 Particle Accelerator Conference—Accelerator Engineering and Technology held in Washington, D. C. on 3 March 1969, Dr. V. P. Sarantsev, head of the Dubna group on electron ring accelerator research, announced that a few days earlier electron rings were successfully extracted.

[4]Various authors, in Lawrence Radiation Laboratory Report No. UCRL-18103, February, 1968 (unpublished).

[5]N. C. Christofilos, in Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958 (United Nations, Geneva, Switzerland, 1958), Vol. 32, pp. 279-290.

[6]N. C. Christofilos, Lawrence Radiation Laboratory, Livermore, Report No. UCRL-5617, 1959 (unpublished).

[7]A parenthetical remark is in order here. The equations for the radial oscillations and radial momentum can be written in the form $r=R+a\sin(\omega t+\varphi)$, $p_r=m\omega a\cos(\omega t+\varphi)$. If the acceleration is slow enough so that $\dot{\omega}/\omega^2 \ll 1$, the adiabatic invariant is $J=m\omega a^2/2$. Thus $a=(2J/m)^{1/2}$, or $a=(2Jc/e)^{1/2}B^{-1/2}$.

[8]N. C. Christofilos, Lawrence Radiation Laboratory, Livermore, Report No. UCID-15377, September, 1968 (to be published).

VLASOV DESCRIPTION OF AN ELECTRON GAS IN A MAGNETIC FIELD*

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An algorithm is presented for obtaining electron-gas stability information from neutral-plasma literature.

In this work we describe some basic properties of a magnetically confined collisionless electron gas in the framework of the nonrelativistic Vlasov equation. The results are pertinent to recent experimental studies[1,2] of this problem. Previous treatments[3,4] of the equilibria and stability of electron gases have been carried out within the framework of macroscopic fluid models. A difficulty with that approach is that it is not straightforward to extend it to include finite temperature.[5] Also, there are instabilities and waves associated with the velocity-space structure of the electron distribution which are not recovered within a macroscopic model. For these reasons we have treated the problem by using the Vlasov equation and obtained (a) a variety of self-consistent simple equilibria including space charge and (b) associated dispersive properties and stability behavior.

The Vlasov equation[6] describes the time evolution of the probable density of electrons $f(\vec{x},\vec{v},t)$ in velocity-configuration phase space $(\vec{x},\vec{v})$ in the absence of collisions and is given by

$$\frac{\partial f}{\partial t}+\vec{v}\cdot\frac{\partial f}{\partial \vec{x}}+\frac{q}{m}\left(\vec{E}+\frac{\vec{v}\times\vec{B}}{c}\right)\cdot\frac{\partial f}{\partial \vec{v}}=0, \quad (1)$$

where

$$\nabla\cdot\vec{E}=4\pi q\int d^3v\, f+4\pi\rho_{\mathrm{ext}},$$

$$\nabla\times\vec{B}=\frac{1}{c}\frac{\partial}{\partial t}\vec{E}+\frac{4\pi q}{c}\int d^3v\,\vec{v}f+\frac{4\pi}{c}\vec{J}_{\mathrm{ext}}, \quad (2)$$

where $q(=-e)$ and m are the charge and mass of the electron, respectively. Thus in Eq. (1) the electric field $\vec{E}(\vec{x}, t)$ and magnetic field $\vec{B}(\vec{x}, t)$ include, in addition to any externally imposed fields, the self-consistent fields produced by the electron charge density and current.

We make the following simplifying assumptions: (i) $\rho_{ext} = 0$, and the external confining field $\vec{B}_0$ is uniform and in the z direction ($\vec{B}_0 = B_0\hat{k}$); (ii) the equilibrium density $n^0(\vec{x}) = \int d^3v f^0$ is cylindrically symmetric about an axis along $\vec{B}_0$, and the equilibrium is uniform in the z direction, with $\partial f^0(\vec{x}, v)/\partial z = 0$ and $\vec{B}_0 \cdot \vec{E}^0(\vec{x}) = 0$; and (iii) the equilibrium self-magnetic field $B_m{}^0(\vec{x})$ (which will result from the azimuthal current associated with a rotating electron gas) is negligible in comparison with B_0. We first examine possible equilibria within the context of these assumptions.

Equilibria.—Setting $\partial f/\partial t = 0$ in the Vlasov equation gives

$$\vec{v}\cdot\frac{\partial f^0}{\partial \vec{x}} + \frac{q}{m}\left(\vec{E}^0 + \frac{\vec{v}\times\vec{B}_0}{c}\right)\cdot\frac{\partial f^0}{\partial \vec{v}} = 0, \tag{3}$$

$$(\partial/\partial\vec{x})\cdot\vec{E}^0 = 4\pi q \int d^3v f^0. \tag{4}$$

Introducing a Cartesian coordinate system with origin on the axis of symmetry and x and y axes perpendicular to $\vec{B}_0$, we see that any function $f^0(H, L, v_z)$ of the constants of motion H, L, and v_z satisfies (3), with

$$H = \tfrac{1}{2}mv_x{}^2 + \tfrac{1}{2}mv_y{}^2 + q\varphi^0(r), \tag{5}$$

and

$$L = m(xv_y - yv_x) - \tfrac{1}{2}mr^2\Omega_c. \tag{6}$$

In (5) and (6) $r = (x^2+y^2)^{1/2}$ and $\Omega_c = -qB_0/mc$. In principle, the choice of $f^0(\varphi^0)$ determines the equilibrium electrostatic potential $\varphi^0(r)$ through (4), i.e.,

$$-\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r}\varphi^0(r) = 4\pi q n^0(r) = 4\pi q \int d^3v f^0. \tag{7}$$

Thus by specifying the functional dependence of f^0 on H and L a variety of equilibrium density profiles $n^0(r)$ [and corresponding $\varphi^0(r)$] can be constructed. Depending on the explicit form of f^0 these equilibria may (or may not) have density profiles which are peaked on the axis of symmetry, or shear in the angular velocity of rotation of the electron gas about the axis.[7] For present purposes we limit considerations to a class of equilibria for which

$$f^0 = f^0(H - \omega_R L, v_z), \tag{8}$$

where $\omega_R = \text{const}$. The meaning of this restriction from the fluid point of view is seen by calculating velocity moments of f^0. Noting that

$$H - \omega_R L = \tfrac{1}{2}m(v_x + \omega_R y)^2 + \tfrac{1}{2}m(v_y - \omega_R x)^2 + \tfrac{1}{2}m\{r^2(\omega_R\Omega_c - \omega_R{}^2) + (2q/m)\varphi^0(r)\}, \tag{9}$$

the mean velocities $\bar{v}_x = (\int d^3v f^0 v_x)(\int d^3v f^0)^{-1}$ and $\bar{v}_y = (\int d^3v f^0 v_y)(\int d^3v f^0)^{-1}$ are given by $\bar{v}_x = -\omega_R y$ and $\bar{v}_y = \omega_R x$ for equilibria of the form (8), corresponding to a mean rigid rotation (with angular velocity ω_R) of the election gas about the axis of symmetry. Even without specifying the form of f^0 explicitly, the choice of ω_R will influence φ^0 and n^0. From (7) [taking $\varphi^0(r=0) = 0$ and denoting $n^0(r=0)$ by n_0] it is clear that if

$$(\omega_R\Omega_c - \omega_R{}^2)/\tfrac{1}{2}\omega_p{}^2 = 1 + \epsilon, \quad 0 < \epsilon \ll 1, \tag{10}$$

where $\omega_p{}^2 = 4\pi n_0 e^2/m$, the equilibrium density profile $n^0(r)$ is flat $[n^0(r) \simeq n_0]$ with $\varphi^0(r) = -(m\omega_p{}^2/4q)r^2$ for a broad interior region of the electron column. The explicit choice of f^0 determines the shape of the boundary and, as we will show, the stability properties. To illustrate this point we consider two particular distributions f^0 [with $\int F(v_z)dv_z = 1$]:

$$f^0 = \frac{1}{2\pi}\left(\frac{m}{2}\right)^{1/2}\frac{n_0}{V_0}\delta\left((H-\omega_R L)^{1/2} - \left[\frac{m}{2}\right]^{1/2}V_0\right)F(v_z), \quad V_0 > 0, \tag{11}$$

and

$$f^0 = \left(\frac{m}{2\pi\theta}\right)n_0 \exp\{-(H-\omega_R L)/\theta\} F(v_z). \tag{12}$$

For Eq. (11), which we call case (a),

$$n^0(r)=n_0,\quad \varphi^0(r)=-(m\omega_p^{\,2}/4q)r^2,\qquad 0<r<R_p, \tag{13}$$

$$n^0(r)=0,\quad \varphi^0(r)=\frac{m\omega_p^{\,2}}{4q}R_p^{\,2}\left(1+2\ln\frac{r}{Rp}\right),\qquad r>R_p, \tag{14}$$

where the column radius R_p is given by

$$R_p=\sqrt{2}\,\frac{V_0}{\omega_p}\left(\frac{\omega_R\Omega_c-\omega_R^{\,2}}{\frac{1}{2}\omega_p^{\,2}}-1\right)^{-1/2} \tag{15}$$

in Eqs. (13) and (14). When ω_R satisfies (10) R_p is large in units of the "Debye length" (V_0/ω_p) and the density is uniform within the column.

Similar conclusions are reached for Eq. (12), which we call distribution (b), with $n^0(r)$ virtually constant until $r\simeq R_p$, only now

$$R_p\sim\frac{(\theta/m)^{1/2}}{\omega_p}\ln\left[\ln\frac{\omega_R\Omega_c-\omega_R^{\,2}}{\frac{1}{2}\omega_p^{\,2}}\right]^{-1}.$$

The distribution (b) has a smooth boundary; $n^0(r)$ drops to zero rapidly $[\sim\exp\{-r^2/4\lambda_D^2\}$, where $\lambda_D^2=(4\pi n_0e^2)^{-1}\theta]$ for $r>R_p\gg\lambda_D$. If (10) is not satisfied with $\epsilon\ll1$, the distribution (b) will be bell shaped, rather than flat with a thin (λ_D) surface thickness.

Thus the two equilibrium rotor frequencies consistent with a flat density profile and a thin boundary, given by Eq. (10) with $\epsilon\to0_+$, are

$$\omega_R=\omega_\pm=\tfrac{1}{2}\Omega_c\{1\pm(1-2\omega_p^{\,2}/\Omega_c^{\,2})^{1/2}\}. \tag{16}$$

Which of the possible equilibria is occupied depends on the experimental preparation of the system.[8]

For low density $(2\omega_p^2/\Omega_c^2\ll1)$, $\omega_+\simeq\Omega_c$ and $\omega_-\simeq(\omega_p^{\,2}/2\Omega_c)$; for Brillouin flow $(2\omega_p^{\,2}=\Omega_c^{\,2})$, $\omega_+=\omega_-=\frac{1}{2}\Omega_c$. We emphasize that the distribution (8) is only rigid rotor in the mean, and does not imply zero temperature.

Perturbation analysis.—We now examine the evolution of small-amplitude electrostatic perturbations about general equilibria of the form (8) when ω_R satisfies condition (10) (i.e., $\omega_R\simeq\omega_+$ or $\omega_R\simeq\omega_-$). Analysis will be restricted to the interior region of the column where

$$n^0(r)=n_0,\quad \varphi^0(r)=-(m\omega_p^{\,2}/4q)r^2. \tag{17}$$

As such, it is only pertinent to interior wave disturbances with (perpendicular) wavelengths short compared with the length scale over which $n^0(r)\simeq n_0$ remains a good approximation. These are "body waves"; we study them to find stability properties which have no analog in the fluid approach. There are of course other possible calculations suggested by the Vlasov technique, in particular the effect of finite temperature on the surface waves and body waves which are found by fluid theory. From (1) the perturbed distribution $f^{(1)}$ and the electric field $\vec{E}^{(1)}$ evolve according to

$$\frac{\partial f^{(1)}}{\partial t}+\vec{v}\cdot\frac{\partial f^{(1)}}{\partial\vec{x}}+\frac{q}{m}\left(\vec{E}^0+\frac{\vec{v}\times\vec{B}^0}{c}\right)\cdot\frac{\partial f^{(1)}}{\partial\vec{v}}=\frac{-q}{m}\vec{E}^{(1)}\cdot\frac{\partial f^0}{\partial\vec{v}}, \tag{18}$$

$$\frac{\partial}{\partial\vec{x}}\cdot\vec{E}^{(1)}=4\pi q\int d^3v\,f^{(1)}$$

in the electrostatic approximation. In (18) we allow the perturbations to generally depend on z. Equation (18) may be solved by integrating along the unperturbed orbits $\vec{x}'(t')$ and $\vec{v}'(t')$ defined by

$$\frac{d\vec{x}'(t')}{dt'}=\vec{v}'(t'),$$

$$\frac{d\vec{v}'(t')}{dt'}=\frac{q}{m}\left\{\vec{E}^0(\vec{x}'(t'))+\frac{\vec{v}'(t')\times\vec{B}_0}{c}\right\}, \tag{19}$$

where $\vec{v}'(t'=t)=\vec{v}$, $\vec{x}'(t'=t)=\vec{x}$, $\vec{x}$ and $\vec{v}$ are independent variables, and $\vec{E}^0(\vec{x}'(t'))=-\nabla\varphi^0$ is given by Eq. (17). The linearized Vlasov equation (18) may be written

$$\frac{d}{dt'}f^{(1)}(\vec{x}'(t'),\vec{v}'(t'),t')=\frac{-q}{m}\vec{E}^{(1)}(\vec{x}'(t'),t')\cdot\frac{\partial}{\partial\vec{v}'}f^0(H-\omega_RL,v_z). \tag{20}$$

Without presenting any of the details here Eq. (20) may be integrated forward in time to determine $f^{(1)}(\vec{x},\vec{v},t)$ in terms of $\vec{E}^{(1)}$ and initial values, and the result substituted into Poisson's equation, $(\partial/\partial\vec{x})\cdot\vec{E}^{(1)}=4\pi q\int f^{(1)}dv$. In the usual manner[6] a Fourier analysis in the position variable $\vec{x}$ and a Laplace transformation in the time variable t leads to a dispersion relation relating the (complex) frequency ω and wave vector $\vec{k}$. For spatial perturbations with azimuthal harmonic number l, the dielectric function $D_l(k_\perp,k_z,\omega)$ is given by (J_n is the Bessel function)

$$D_l = 1 + \sum_{n=-\infty}^{n=+\infty} \frac{4\pi e^2}{mk^2} \int d^3v\, J_n^{\,2}\left(\frac{k_\perp v_\perp}{|\omega_+ - \omega_-|}\right) \left\{ k_z \frac{\partial}{\partial v_z} + \frac{n(\omega_+ - \omega_-)}{v_\perp} \frac{\partial}{\partial v_\perp} \right\} f^0(v_\perp^{\,2}, v_z) \times [\omega - l\omega_R - k_z v_z - n(\omega_+ - \omega_-)]^{-1}, \qquad (21)$$

where $k_\perp = (k_x^{\,2} + k_y^{\,2})^{1/2}$ and $k^2 = k_z^{\,2} + k_\perp^{\,2}$. The variable $v_\perp$ defined by

$$v_\perp^{\,2} = (2/m)(H - \omega_R L) \simeq (v_x + \omega_R y)^2 + (v_y - \omega_R x)^2 \qquad (22)$$

has been introduced in Eq. (21)[9] and $\int d^3v \equiv 2\pi \times \int_{-\infty}^{\infty} dv_z \int_0^\infty v_\perp dv_\perp$. The rigid-rotor frequency ω_R appearing in (21) is either ω_+ or ω_- according as the equilibrium distribution f^0 corresponds to rotation with ω_+ or ω_-. It is clear that the modes determined from the dispersion relation $D_l(k_\perp, k_z, \omega) = 0$ will differ for the two types of equilibria.[10]

The striking feature of the dielectric function (21) is that it is identical to the corresponding results for a neutral plasma in an external magnetic field,[11] using the algorithm

$$\omega \to \omega - l\omega_R, \quad \Omega_c \to \omega_+ - \omega_- = \Omega_c(1 - 2\omega_p^{\,2}/\Omega_c^{\,2})^{1/2} \qquad (23)$$

in this latter reference (and taking the positive ions infinitely massive). Consequently, any electrostatic mode or velocity-space instability which involves a single species, for a plasma, has its analog in a pure electron gas in situations where the present analysis is applicable, and a large body of neutral-plasma results may be applied virtually intact.

For example, when $k_z = 0$ an equilibrium distribution of the form (12) when substituted in (21) gives the analog of the Bernstein-mode dispersion relation.[12] As such the associated modes are pure oscillatory ($\mathrm{Im}\,\omega = 0$) with $\mathrm{Re}\,\omega - l\omega_R = n[1 + \alpha(n)](\omega_+ - \omega_-)$, $n = \pm 1, \pm 2$, where

$$\alpha(n) \simeq \frac{\omega_p^{\,2}}{(k_\perp^{\,2}\theta/m)} \exp\left\{ -\frac{k_\perp^{\,2}\theta/m}{(\omega_+ - \omega_-)^2} \right\} \times I_n\left(\frac{k_\perp^{\,2}\theta/m}{(\omega_+ - \omega_-)^2} \right),$$

when $\alpha(n) \ll 1$. It might also be noted in relation to the distribution (12) that in the high-density limit $2\omega_p^{\,2} = \Omega_c^{\,2}$ (i.e., $\omega_+ - \omega_- \to 0_+$, $\omega_R = \omega_+ = \omega_- = \frac{1}{2}\Omega_c$) corresponding to Brillouin flow, the dielectric function (21) gives the dispersion relation

$$(\omega - \tfrac{1}{2}l\Omega_c)^2 \simeq \omega_p^{\,2}(1 + 3k_\perp^{\,2}\lambda_D^{\,2} + \cdots)$$

for wavelengths long compared with the Debye length $\lambda_D = (\theta/4\pi n_0 e^2)^{1/2}$; thus the pure electron gas exhibits plasma oscillations with the familiar thermal corrections.

In contrast, an equilibrium distribution of the form (11) when substituted in (21) (again with $k_z = 0$) is unstable ($\mathrm{Im}\,\omega > 0$) provided the electron density is sufficiently high. In particular, making the replacement (23) in the instability condition for the corresponding plasma problem,[13,14] we see that the electron gas is unstable provided $[\omega_p^{\,2}/(\omega_+ - \omega_-)^2] \gtrsim 6.62$, i.e., $(\omega_p^{\,2}/\Omega_c^{\,2}) \gtrsim 0.46$, which is a lower density threshold than in the plasma case. Moreover the threshold is slightly below the density corresponding to Brillouin flow ($2\omega_p^{\,2} = \Omega_c^{\,2}$) of the electron gas. The distribution (11) and other distributions of the loss-cone form[13] are especially revelant for mirror-confined electron gases.

As a final example, if $2\pi\int_0^\infty dv_\perp\, v_\perp f^0(v_\perp^{\,2}, v_z)$ corresponds to two counterstreaming (in v_z) electron streams, as might be the case for an electron beam reflecting from a magnetic mirror and passing through the incoming beam, the familiar two-stream instability[6] emerges from (21). In particular, for $2\pi\int_0^\infty dv_\perp\, v_\perp f^0(v_\perp^{\,2}, v_z) = \frac{1}{2}n_0\{\delta(v_z - u) + \delta(v_z + u)\}$ we have (for $k_\perp = 0$)

$$\omega^2 = \tfrac{1}{2}[2k_z^{\,2}u^2 + \omega_p^{\,2} \pm \omega_p(\omega_p^{\,2} + 8k_z^{\,2}u^2)^{1/2}],$$

which gives instability for $k_z^{\,2} < \omega_p^{\,2}/u^2$ with maximum growth rate $(\mathrm{Im}\,\omega)_{\max} = \omega_p/2\sqrt{2}$.

The above examples form only a small subsection of the vast array of (body) instabilities and waves characteristic of an electron gas. It is apparent that a detailed survey and classification is needed in this regard.

In conlcusion we note that the terminology "electron plasma" must be appropriate when discussing a pure electron gas since the dispersive properties and the phenomenon of Debye shielding are quite analogous.

The authors have profited from exhaustive discussions on this subject with Professor Alvin Trivelpiece.

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[1]A. W. Trivelpiece, R. E. Pechacek, and C. A. Kapentanakos, Phys. Rev. Letters 21, 1436 (1968).

[2]University of California Lawrence Radiation Laboratory Report No. UCRL-18103, 1968 (unpublished).

[3]J. D. Daugherty and R. H. Levy, Phys. Fluids 10, 155 (1967).

[4]R. H. Levy, Phys. Fluids 11, 920 (1968).

[5]K. V. Roberts and J. B. Taylor, Phys. Rev. Letters 8, 197 (1962).

[6]D. Montgomery and D. A. Tidman, Plasma Kinetic Theory (McGraw-Hill Book Company, Inc., New York, 1964), pp. 51-59, 177.

[7]A simple equilibrium which has both shear in angular velocity of rotation and a density profile which is peaked off axis is $f^0 = AL^2 \exp\{-(H-\omega_R L)/\theta\} F(v_z)$, where A and θ are positive constants, $\int F(v_z) dv_z = 1$, and $\omega_R = \text{const}$.

[8]Equilibria with flat density profiles may also be constructed with arbitary mean rigid-rotor frequency $\bar{\omega}_R$, $\omega_- \lesssim \bar{\omega}_R \lesssim \omega_+$, by taking f^0 to be the linear combination $f^0 = \alpha_+ f_+{}^0 (H-\omega_R L)|_{\omega_R \simeq \omega_+} + \alpha_- f_-{}^0 (H-\omega_R L)|_{\omega_R \simeq \omega_-}$, where $\alpha_+ + \alpha_- = 1$ and $\bar{\omega}_R = \alpha_+ \omega_+ + \alpha_- \omega_-$.

[9]The final equality in (22) follows directly from Eqs. (9), (10), and (17).

[10]The dielectric function (21) is generalized to include the equilibria described in footnote 8 by replacing f^0 by $\alpha_i f_i{}^0$, ω_R by ω_i, and summing over $i = +,-$ in Eq. (21).

[11]E. G. Harris, J. Nucl. Energy C2, 138 (1961).

[12]I. B. Bernstein, Phys. Rev. 109, 10 (1958).

[13]R. A. Dory, G. E. Guest, and E. G. Harris, Phys. Rev. Letters 14, 132 (1965).

[14]F. W. Crawford and J. A. Tataronis, J. Appl. Phys. 36, 2930 (1965).