QUANTIZATION CONDITIONS FOR REGGE INTERCEPTS AND HADRON MASSES*

M. Ademollo†
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts

and

G. Veneziano‡ and S. Weinberg
Laboratory for Nuclear Science and Department of Physics,
Massachusetts Institute of Technology, Cambridge, Massachusetts
(Received 4 December 1968)

We show that a reasonable generalization of the beta-function formula to the reaction $\pi+A\rightarrow B+C$, combined with Adler's self-consistency condition, implies that trajectories of opposite "normality" that can be connected by pion emission must have the same slope, and that their intercepts must differ by a half-odd integer. The agreement with experiment seems very good.

Lovelace[1] has recently derived a number of relations among meson widths and masses by applying the Adler "self-consistency condition"[2] to a model of $\pi^-\pi$ scattering based on the beta-function formula.[3] In this Letter we shall extend Lovelace's arguments to a general scattering process $\pi+A\rightarrow B+C$ and will show that they lead to a remarkable quantization of Regge intercepts and hadron masses.

We are going to require that the S matrix for the process $\pi+A\rightarrow B+C$ vanishes when the pion four-momentum vanishes; so we are only interested in those invariant amplitudes for which this condition is nontrivial. Such amplitudes only receive contributions from Regge trajectories in the s channel whose normality[4] is the same as that of A and an s-wave pion, and hence opposite to that of A, and similarly from trajectories in the t and u channels whose normalities are opposite to those of B and C, respectively. We will assume that these amplitudes therefore receive contributions from only one family of linear trajectories (i.e., one or more degenerate leading trajectories and infinitely many parallel daughters) in each channel, and will call the leading trajectories in these families $\alpha_X(s)$, $\alpha_Y(t)$, and $\alpha_Z(u)$. In the spirit of Ref. 3, we then expect that each of these amplitudes may be represented as a sum of terms of the form

$$g\frac{\Gamma(k+J_A-\alpha_X(s))\,\Gamma(l+J_B-\alpha_Y(t))}{\Gamma(n+J_A+J_B-\alpha_X(s)-\alpha_Y(t))},\qquad(1)$$

plus similar terms which depend on s,u and t,u. Here k, l, and n are constants (with $n\leq l+k$) which may be determined by noting that the lowest resonances in the s and t channels which can contribute to this term have spins $k+J_A$ and $l+J_B$, so that k and l are integers, and that its asymptotic behavior as $|s|\rightarrow\infty$ with t fixed is $s^{\alpha(t)-\Delta}$, where $\Delta=J_B+n-k$. But Δ is a helicity difference which can differ from J_B by at most an integer (in fact, the interesting amplitudes have $\Delta=J_B$), so n is also an integer.

The requirement that the S matrix vanish when $p_\pi{}^\mu=0$ may in some cases be achieved by a cancellation of s,t terms of the form of Eq. (1) with the s,u and t,u terms. However, we have found that the interesting invariant amplitude always contains at least one s,t term which must survive any possible cancellation (if the whole amplitude is not to vanish) in the reactions[5] with $A=C=\pi$, $B=A_1,H,g$, etc.; $A=\pi$, $B=\rho$, $C=\sigma$; $A=C=N$, $B=\pi$; and $A=B=N$, $C=\pi$. Lovelace's work[1] has already shown that this is true for $A=B=C=\pi$. (In all these examples it turns out that this term has $n=1$, but we do not expect this always to be the case.) The problem of determining for which amplitudes and n values cancellations are possible at $p_\pi{}^\mu=0$ is both important and difficult, and will be left for future work. For the present we will simply assume that for any pair of particles A,B we can choose C so that at least one of the invariant amplitudes has an s,t term which is not cancelled at $p_\pi{}^\mu=0$ by other terms. The vanishing of this term then requires that the Γ function in the denominator blows up when $p_\pi{}^\mu=0$, i.e., when $s=m_A{}^2$, $t=m_B{}^2$, $u=m_C{}^2$. Letting n_{AB} be the largest n value in any such term, we conclude that

$$\alpha_X(m_A{}^2)+\alpha_Y(m_B{}^2)=J_A+J_B+N_{AB},$$

$$N_{AB}=n_{AB}\text{ or }n_{AB}+1\text{ or }\cdots.\qquad(2)$$

In the special case with $A=B$ and $X=Y$, Eq. (2)

teils us that

$$\alpha_X(m_A{}^2)=J_A+\tfrac{1}{2}N_{AA}. \quad (3)$$

Inserting this back into Eq. (2) we see that

$$N_{AA}+N_{BB}=2N_{AB}. \quad (4)$$

Since A and B can be any two particles, Eq. (4) shows that the N_{AA} are either all even or all odd integers. (It was just to get this result that we had to consider a completely general process $\pi A \to BC$.) Both Lovelace and experiment agree that when $A=\pi$ and $X=\rho$ the integer $N_{\pi\pi}$ in Eq. (3) is unity; hence we conclude that N_{AA} is always an odd integer.

The power of Eq. (3) is revealed when we make use of the fact that the external particle A is itself on a linear Regge trajectory to determine $m_A{}^2$. Eq. (3) then becomes

$$\alpha_X(0)+(\alpha_X{}'/\alpha_A{}')[J_A-\alpha_A(0)]=J_A+\tfrac{1}{2}N_{AA}. \quad (5)$$

This must also be true for the particle A of spin J_A on the parallel daughter trajectory which lies one unit below the A trajectory; so $\alpha_X{}'/\alpha_A{}'$ equals $\frac{1}{2}(N_{AA}-N_{aa})$, which is necessarily an integer. If we had considered an external particle lying on a trajectory in the X family, we would similarly have found that $\alpha_A{}'/\alpha_X{}'$ must be an integer. This integer must then be unity; so

$$\alpha_A{}'=\alpha_X{}'; \quad (6)$$

and using this in (5) gives

$$\alpha_X(0)-\alpha_A(0)=\tfrac{1}{2}N_{AA}. \quad (7)$$

Our conclusion then is that whenever particles on one trajectory can decay by pion emission into particles of opposite normality on another trajectory, the two trajectories must have the same slope and must have intercepts which differ by a half-odd integer.

The comparison of these predictions with the data[6] for four pairs of trajectories of opposite normality is shown in Table I. Evidently there is a real tendency for corresponding trajectories to have the same slope, and an even stronger tendency for their intercepts to differ by $\frac{1}{2}$. (The N_γ trajectory does not seem to compare well with the Δ trajectory, but this may be because its Chew-Frautschi plot is not straight.) Presumably the reason that we always get an intercept difference of $\frac{1}{2}$ rather than $\frac{3}{2}$, $\frac{5}{2}$, etc., is because we are comparing the leading trajectories for each set of intrinsic quantum numbers.

Table I. Observed slopes and intercepts of pairs of trajectories of opposite normality which can be connected by single-pion emission.

Trajectory	$\alpha(s)$	Difference of intercept
$\rho(f)$	$0.46+0.85s$	
$\pi(A_1)$	$-0.02+0.85s$	0.48
Δ	$0.15+0.90s$	
N	$-0.39+1.01s$	0.54
K^*	$0.34+0.83s$	
K	$-0.19+0.77s$	0.53
$Y_1^*(1385)$	$-0.25+0.91s$	
{Σ	$-0.77+0.90s$	0.52
{Λ	$-0.68+0.95s$	0.43

Mandelstam[7] has argued that a model with linearly rising trajectories must have all trajectories parallel. This is not in outrageous disagreement with the data (see Table I); so we may tentatively suppose that all trajectories have the same slope as the ρ trajectory, which [with $m_\pi=0$ and $\alpha_\rho(0)=\frac{1}{2}$] is just $(2m_\rho{}^2)^{-1}$. Our result for the Regge intercepts can then be converted into a great many mass formulas, such as

$$m_{A_1}{}^2=2m_\rho{}^2;\quad m_\Delta{}^2=m_N{}^2+m_\rho{}^2;$$

$$m_{K^*}{}^2=m_K{}^2+m_\rho{}^2;\quad m_B{}^2=m_\omega{}^2+m_\rho{}^2;$$

$$m_{A_2}{}^2=3m_\rho{}^2+m_\eta{}^2;\quad m_f{}^2=3m_\rho{}^2;\quad \text{etc.},$$

all of which seem to agree pretty well with experiment.

We close with a few remarks:

(1) The great power of the condition imposed by the partial conservation of axial-vector current for one soft pion should perhaps not be so surprising, for Mandelstam has already shown[8] that this condition implies the current algebra results for two soft pions.

(2) The mass formula $m_{K^*}{}^2-m_\rho{}^2=m_K{}^2-m_\pi{}^2$ was derived by Lovelace[1] by requiring that the $K\pi$ scattering amplitude vanish for $p_K{}^\mu=0$ as well as for $p_\pi{}^\mu=0$. However this derivation leads to an inconsistency unless the masses are SU(3) degenerate. Our derivation uses soft pions only.

(3) We have not used the conditions of crossing symmetry and the absence of exotic (e.g., $T=2$) resonances, except that these conditions play a role in ruling out possible cancellations at $p_\pi{}^\mu=0$ in processes like π-π scattering. However the absence of $T=2$ resonances ensures that the π-π

scattering amplitude at $s=u=m_\pi^2$ and $t=0$ is pure $T=0$ in the t channel, because at this point the $T=2$ part of the amplitude arises solely from the background s,u term, which vanishes at $s=u=t=m_\pi^2$ for all t. This implies that the pion expectation value of the σ term $[A^0, \partial_\mu A^\mu]$ has no $T=2$ part and also explains why Lovelace found[1] a scattering-length ratio a_0/a_2 close to $-\frac{7}{2}$.

(4) The Lovelace-Veneziano model for π-π scattering predicts that the ρ and σ mesons do not saturate the chiral commutation relations but only account for a fraction $2/\pi$ of the required value of the sum over states. (This is for $m_\pi=0$; for the actual m_π this ratio is close to $\frac{2}{3}$.) It is therefore a mystery why Lovelace gets essentially the same value for the ratio of the ρ and σ widths as was calculated[9] by using the chiral commutation relations, throwing away all other states. In particular, this current-algebra calculation gave for Γ_ρ the "KSFR value" (about 135 MeV), while the Lovelace-Veneziano model predicts that Γ_ρ is less than this by the factor $2/\pi$, or $\Gamma_\rho \approx 90$ MeV.

We are grateful for discussions with S. Fubini.

---

*Work supported in part through funds provided by the U. S. Atomic Energy Commission under Contract No. AT(30-1) 2098, by the U. S. Office of Naval Research under Contract No. Nonr-1866 (55), and by the U. S. Air Force under Contract No. 49 (638)-1380.

†On leave of absence from the Institute di Fisica dell' Università, Firenze, Italy.

‡On leave of absence from the Weizmann Institute of Science, Rehovoth, Israel.

[1]C. Lovelace, CERN Report No. TH950 (to be published).

[2]S. Adler, Phys. Rev. 137, B1022 (1965).

[3]G. Veneziano, Nuovo Cimento 57A, 190 (1968); also see S. Fubini, in "Comments on Nuclear and Particle Physics" (to be published); G. Veneziano, to be published; M. A. Virasoro, University of Wisconsin Report No. COO-209 (to be published). For π-π scattering see J. Shapiro and J. Yellin, University of California Radiation Report No. UCRL 18500 (to be published).

[4]Normality is defined as parity times $(-)^J$ for bosons or times $(-)^{J-\frac{1}{2}}$ for fermions.

[5]For π-N scattering we have used the results of Virasoro, Ref. 3.

[6]These Regge slopes and intercepts are determined by using observed masses of the two lowest particles in Chew-Frautschi plots, with particles assigned to trajectories as suggested by Ref. 1 for ρ. For Δ and N, see V. Barger and D. Cline, Phys. Rev. 155, 1792 (1967). For K and K^*, see B. French in the Proceedings of the Fourteenth International Conference on High Energy Physics Vienna, Austria, 1968 (European Organization for Nuclear Research, Geneva, Switzerland, 1968), p. 97. For Σ, Λ, and Y_1^*, see P. D. B. Collins and E. J. Squires, Regge Poles in Particle Physics (Springer-Verlag, Berlin, Germany, 1968), p. 199.

[7]S. Mandelstam, Phys. Rev. 166, 1539 (1968), and to be published.

[8]S. Mandelstam, Phys. Rev. 168, 1884 (1968).

[9]F. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968); S. Weinberg, to be published.

REGGE CUTS AND FINITE ENERGY SUM RULES*

R. J. Rivers
The Enrico Fermi Institute, The University of Chicago, Chicago, Illinois
(Received 11 November 1968)

Including Regge-cut amplitudes in finite-energy sum rules for inelastic scattering we examine the possibilities that either (a) they are built up by part of the direct-channel resonance contributions, in which case they can significantly alter some of the bootstrap predictions, or (b) they are built up by a nonresonant background, in which case mesons with $Y=2$ are required that couple to the KK system.

There has recently been much discussion of finite-energy sum rules[1] (FESR). The FESR for an invariant amplitude $A(s,t)$ are derived from the superconvergence of $A(s,t)-R(s,t)$, the amplitude minus its asymptotic Regge behavior $R(s,t)$, and state that

$$\int^S s'^n \,\mathrm{Im}A(s',t)ds' \approx \int^S s'^n \,\mathrm{Im}R(s',t)ds'. \quad (1)$$

Equation (1) is valid for $s \gtrsim s_0$, where s_0 is the lowest energy for which the Regge representation of the amplitude is assumed valid.

For inelastic scattering it has been suggested that $\mathrm{Im}A(s,t)$ and $\mathrm{Im}R(s,t)$ be approximated by $\mathrm{Im}A_{\mathrm{res}}(s,t)$ and $\mathrm{Im}R_{\mathrm{pole}}(s,t)$, respectively, where $A_{\mathrm{res}}(s,t)$ is the sum of direct channel resonances and $R_{\mathrm{pole}}(s,t)$ is the sum of the t-channel Regge-pole amplitudes. In a narrow-width approximation with indefinitely rising trajectories the FESR then become "bootstrap" equa-