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DISTORTIONLESS PROPAGATION OF LIGHT THROUGH AN OPTICAL MEDIUM*

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The equations describing the propagation of optical radiation through a medium of two-level atoms are presented in a way that emphasizes their relation to Poynting's theorem. Two classes of solutions which propagate without distortion are discussed. These solutions, together with the well-known hyperbolic secant solution, are analogous to the three possible types of motions of a simple pendulum.

Consider a medium consisting of N two-level atoms per unit volume imbedded in a homogeneous dielectric which is characterized by an index of refraction $\eta = (\epsilon_0\mu_0)^{1/2}$. It will be assumed that the magnetic permeability μ_0 is approximately equal to unity while the dielectric constant ϵ_0 may differ from unity. Conservation of energy for light propagating through such a medium is expressed by Poynting's theorem[1]

$$\partial u(\vec{x}, t)/\partial t + \nabla\cdot\vec{S}(\vec{x}, t) = -\vec{E}(\vec{x}, t)\cdot\partial\vec{P}(\vec{x}, t)/\partial t, \tag{1}$$

where $u(\vec{x}, t)$ is the energy density of the electromagnetic field, $\vec{S}(\vec{x}, t)$ is its Poynting vector, and $\vec{E}(\vec{x}, t)$ is the electric-field strength.

For the case of a quasimonochromatic electromagnetic field linearly polarized in the $\hat{e}_1$ direction and propagating in the $\hat{e}_3$ direction, one may write

$$\vec{E}(\vec{x}, t) = \hat{e}_1\mathcal{E}(z, t)\cos(\omega t - kz), \tag{2a}$$

$$u(\vec{x}, t) = (\epsilon_0/4\pi)\mathcal{E}^2(z, t)\cos^2(\omega t - kz), \tag{2b}$$

$$\vec{S}(\vec{x}, t) = (c/4\pi)(\epsilon_0/\mu_0)^{1/2}\hat{e}_3\mathcal{E}^2(z, t)\cos^2(\omega t - kz). \tag{2c}$$

A more general expression for the field could be used; however, the solutions presented below can be written in the form of Eq. (2a).

The state of a two-level atom may be expressed as

$$\Psi(\vec{x}, t) = a(t)\psi_a(\vec{x}) + b(t)\psi_b(\vec{x}), \tag{3}$$

where ψ_a and ψ_b are the eigenfunctions of the unperturbed atomic Hamiltonian, which correspond to the eigenvalues $\frac{1}{2}\hbar\Omega$ and $-\frac{1}{2}\hbar\Omega$, respectively. Alternately, the state of the atom may be represented by the real variables X, Y, and Z, which are defined according to[2]

$$(X - iY)e^{-i(\omega t - kz)} \equiv 2ab^*, \tag{4a}$$

$$Z \equiv aa^* - bb^*. \tag{4b}$$

These variables satisfy the condition $X^2 + Y^2 + Z^2 = 1$ when $aa^* + bb^* = 1$.

For an inhomogeneously broadened medium which is characterized by a normalized distribution function $g(\Delta)$, the dipole moment per unit volume would be

$$\vec{P}(\vec{x}, t) = N\vec{\mu}_{ab}\int_{-\infty}^{\infty}[X(z,t,\Delta)\cos(\omega t-kz)-Y(z,t,\Delta)\sin(\omega t-kz)]g(\Delta)d\Delta, \tag{5}$$

where $X(z, t, \Delta)$ and $Y(z, t, \Delta)$ are the variables defined in Eq. (4a) which refer to an atom with transition frequency $\Omega = \omega + \Delta$. Substituting from Eqs. (2a), (2b), and (2c) and Eq. (5) into Eq. (1) and assuming that X, Y, and $\mathcal{E}$ are slowly varying, one obtains

$$\frac{c}{\eta}\frac{\partial\mathcal{E}}{\partial z}(z,t)+\frac{\partial\mathcal{E}}{\partial t}(z,t)=\frac{2\pi N\mu\omega}{\eta^2}\int_{-\infty}^{\infty}Y(z,t,\Delta)g(\Delta)d\Delta, \tag{6a}$$

$$\left(\frac{c}{\eta}k-\omega\right)\mathcal{E}(z,t)=\frac{2\pi N\mu\omega}{\eta^2}\int_{-\infty}^{\infty}X(z,t,\Delta)g(\Delta)d\Delta, \tag{6b}$$

after equating the coefficients of $\cos(\omega t-kz)$ and $\sin(\omega t-kz)$ separately. When an atomic system which has a transition frequency $\Omega = \omega + \Delta$ experiences an electric field of the form of Eq. (2a), the variables defined by Eqs. (4a) and (4b) evolve according to[2]

$$\dot{X}(z,t,\Delta) = -\Delta Y(z,t,\Delta), \tag{7a}$$

$$\dot{Y}(z,t,\Delta) = \Delta X(z,t,\Delta) + \hbar^{-1}\mu\mathcal{E}(z,t)Z(z,t,\Delta), \tag{7b}$$

$$\dot{Z}(z,t,\Delta) = -\hbar^{-1}\mu\mathcal{E}(z,t)Y(z,t,\Delta). \tag{7c}$$

The counter-rotating-wave approximation has been made in the derivation of Eqs. (7a), (7b), and (7c). It can be seen from Eqs. (7a) and (7b) that $X(z, t, \Delta)$ is an odd function of Δ and therefore if $g(\Delta)$ is symmetric about $\Delta = 0$, Eq. (6b) reduces with $\omega = ck/\eta$.

McCall and Hahn[3] have studied pulse solutions of equations similar to Eqs. (6a) and (6b) and (7a), (7b), and (7c) in great detail. This paper presents additional solutions of the equations which should be useful in understanding the propagation of long pulses or cw light through matter. It has been found that there are two simultaneous solutions of Eqs. (6a) and (6b) and Eqs. (7a), (7b), and (7c) which can be expressed in terms of Jacobian elliptic functions.[4] The first type of solution may be written as

$$\mathcal{E}(z,t)=\mathcal{E}_0\,\mathrm{dn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right);\lambda\right], \tag{8a}$$

$$X(z,t,\Delta)=\frac{2\Delta\tau}{[(\lambda^2-\Delta^2\tau^2)^2+4\Delta^2\tau^2]^{1/2}}\,\mathrm{dn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right);\lambda\right], \tag{8b}$$

$$Y(z,t,\Delta)=\frac{2\lambda^2}{[(\lambda^2-\Delta^2\tau^2)^2+4\Delta^2\tau^2]^{1/2}}\,\mathrm{sn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right);\lambda\right]\mathrm{cn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right);\lambda\right], \tag{8c}$$

$$Z(z,t,\Delta)=\frac{-(2-\lambda^2+\Delta^2\tau^2)+2\,\mathrm{dn}^2\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right);\lambda\right]}{[(\lambda^2-\Delta^2\tau^2)^2+4\Delta^2\tau^2]^{1/2}}, \tag{8d}$$

where the definition

$$1/\tau \equiv \mu\mathcal{E}_0/2\hbar$$

has been made. Physically, the above solution corresponds to a cw electromagnetic wave whose amplitude is modulated periodically [see Fig. 1(a)] with a period

$$T = 2\tau K(\lambda), \tag{9}$$

where $K(\lambda)$ is the complete elliptic integral of the first kind. This amplitude modulation is similar to the optical nutation effect[5] in that it results from the atoms periodically absorbing energy from one part of the wave and then returning the energy to an adjacent part. The amplitude-modulation function of Eq. (8a) is of special interest because it propagates without distortion. For the case of exact resonance, the angle through which the vector (X, Y, Z) is turned during one period is

$$\varphi = \int_{-\tau K}^{+\tau K}\frac{\mu\mathcal{E}(z,t)}{\hbar}dt = 2\pi. \tag{10}$$

This fact suggests that the waveform of Eq. (8a) is the cw analog of McCall and Hahn's hyperbolic secant pulse.[6] In fact, as the modulus λ approaches unity, the period of $\mathcal{E}$ approaches infinity and the elliptic solutions go into McCall and Hahn's solution, i.e.,

$$\lim_{\lambda\to 1} \mathcal{E}_0 \operatorname{dn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right] = \mathcal{E}_0 \operatorname{sech}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right)\right]. \tag{11}$$

In the limit that λ goes to zero, the amplitude-modulation function becomes constant:

$$\lim_{\lambda\to 0} \mathcal{E}_0 \operatorname{dn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right] = \mathcal{E}_0, \tag{12}$$

and the solution (8b), (8c), and (8d) becomes

$$X = \frac{2}{(\Delta^2\tau^2+4)^{1/2}}, \tag{13a}$$

$$Y = 0, \tag{13b}$$

$$Z = \frac{-\Delta\tau}{(\Delta^2\tau^2+4)^{1/2}}. \tag{13c}$$

Equations (13a) and (13b) describe a polarization which is locked in phase with respect to the applied field [see Eq. (5)]. This solution would be an optical analogy of spin locking.[7]

An expression for the velocity of propagation of the amplitude-modulation function may be found by substituting Eqs. (8a) and (8c) into Eq. (6a). One finds that the reciprocal of the velocity

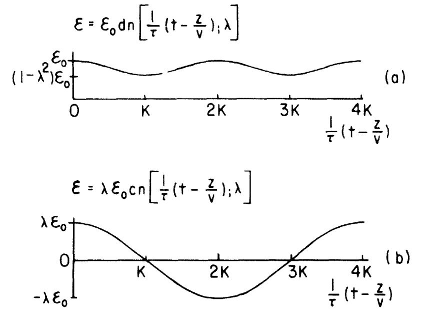

FIG. 1. Graphs of the two types of amplitude modulation for case $\lambda = 0.8$.

is

$$\frac{1}{v} = \frac{\eta}{c} + \frac{2\pi N\omega\mu^2\tau^2}{\hbar c\eta} \times \int_{-\infty}^{\infty} \frac{g(\Delta)d\Delta}{[(\lambda^2-\Delta^2\tau^2)^2+4\Delta^2\tau^2]^{1/2}}. \tag{14}$$

If $g(\Delta)$ is not symmetric about $\Delta = 0$, a dispersion equation of the form $k = k(\omega)$ can be derived by substituting Eqs. (8a) and (8b) into Eq. (6b). The power transported by the amplitude-modulated wave of Eq. (8a), averaged over a period, is

$$\frac{1}{2\tau K}\int_{-\tau K}^{+\tau K} \frac{c}{8\pi}\mathcal{E}^2(z,t)dt = \frac{c}{8\pi}\mathcal{E}_0^{\,2}\frac{E(K(\lambda))}{K(\lambda)}, \tag{15}$$

where $E(K)$ is the complete elliptic integral of the second kind.

The second class of solutions of Eqs. (6a) and (6b) and (7a), (7b), and (7c) may be written as

$$\mathcal{E}(z,t) = \lambda\mathcal{E}_0 \operatorname{cn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right], \tag{16a}$$

$$X(z,t,\Delta) = \frac{2\Delta\tau\lambda}{[(1-\Delta^2\tau^2)^2+4\Delta^2\tau^2\lambda^2]^{1/2}} \operatorname{cn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right], \tag{16b}$$

$$Y(z,t,\Delta) = \frac{2\lambda}{[(1-\Delta^2\tau^2)^2+4\Delta^2\tau^2\lambda^2]^{1/2}} \operatorname{sn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right] \operatorname{dn}\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right], \tag{16c}$$

$$Z(z,t,\Delta) = \frac{-(1+\Delta^2\tau^2) + 2\operatorname{dn}^2\left[\frac{1}{\tau}\left(t-\frac{z}{v}\right); \lambda\right]}{[(1-\Delta^2\tau^2)^2+4\Delta^2\tau^2\lambda^2]^{1/2}}. \tag{16d}$$

Physically, this solution corresponds to a periodic amplitude modulation [see Fig. 1(b)] with a period given by

$$T = 4\tau K(\lambda). \tag{17}$$

For the case of exact resonance, the vector (X, Y, Z) is turned through the angle

$$\varphi = \int_{-2\tau K}^{+2\tau K} \frac{\mu\mathcal{E}}{\hbar}(z,t)dt = 0$$

during one period. These solutions approach the hyperbolic secant solutions as λ goes to unity. The power transported by the waves, averaged over a period, is equal to

$$\frac{1}{4K\tau}\int_{-2K\tau}^{+2K\tau}\frac{c}{8\pi}\mathcal{E}^2(z,t)dt=\frac{c}{8\pi}\mathcal{E}_0^2\left\{\frac{E(K(\lambda))}{K(\lambda)}+(\lambda^2-1)\right\}. \tag{18}$$

The reciprocal velocity of the modulation amplitude is given by

$$\frac{1}{v}=\frac{\eta}{c}+\frac{2\pi N\omega\mu^2\tau^2}{\hbar c\eta}\int_{-\infty}^{\infty}\frac{g(\Delta)d\Delta}{[(1-\Delta^2\tau^2)^2+4\Delta^2\tau^2\lambda^2]^{1/2}}. \tag{19}$$

The solutions presented above[8] correspond to a simple undamped model. In a real solid, relaxation effects would interfere with the coherent absorption and emission of radiation which is necessary for distortionless propagation of light. In order to observe the propagation effects described here, it would be necessary to have relaxation times long compared with the period of the amplitude-modulation function. From Eq. (9) and the definition of τ it is seen that rapid amplitude modulation can be achieved in the presence of strong fields $\mathcal{E}_0=2\hbar/\mu\tau$.

I wish to thank Professor S. R. Hartmann for several helpful and stimulating discussions.

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*Work supported wholly by the Joint Services Electronics Program (U.S. Army, U.S. Navy, and U.S. Air Force) under Contract No. DA-28-043 AMC-00099 (E).

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DETECTION OF SINGLE QUANTIZED VORTEX LINES IN ROTATING He II *

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We have utilized the trapping of electrons on quantized vortex lines in rotating He II to permit detection of single lines in order to study the appearance and disappearance of the first few lines in a capillary rotating about its axis. It seems likely that an extension of the method will permit study of the spatial distribution of vortices in rotating vessels of He II.

It is widely believed that superfluid velocity fields are irrotational everywhere except at singular lines around which the circulation is quantized in units of Planck's constant divided by m, the mass of a helium atom.[1] For a cylindrical bucket of helium, rotating about its symmetry axis, theory[2] predicts that as the angular velocity ω is raised from zero, the helium should remain at rest until a first critical angular velocity (which depends on the bucket radius R) is reached. Above this, the equilibrium state of the system has a single vortex line along the axis; at higher angular velocities more vortices appear. When their number N is large it approaches $\omega m R^2/\hbar$.

The most successful experiment carried out under conditions where only a few vortices should be present is probably that of Hess and Fairbank.[3] They verified one prediction of the vortex picture by finding the angular momentum of a rotating capillary to be lower than that corresponding to rigid-body rotation. They did not, however, resolve any step structure associated with the appearance of single vortices. In rotating He II electrons become trapped on structures which inhibit their motion perpendicular to the axis of rotation but which allow them to be mobile paral-