ues of P which would be difficult to measure optically. Finally, various applications of this method might be used to set up magnetometers or gyrometers.

[1]See, for example, A. Abragam, The Principles of Nuclear Magnetism (Oxford University Press, London, England, 1961).

[2]See, for example, references in C. Cohen-Tannoudji and A. Kastler, Progress in Optics (North-Holland Publishing Co., Amsterdam, The Netherlands, 1966), Vol. V.

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[6]As long as this field is much smaller than the width (about 5 μG) of the level-crossing dispersion resonance curve.

[7]Since $\vec{h}$ is perpendicular to $\vec{H}_1 \cos\omega t$, it does not affect the signal detected by the ^{87}Rb atoms.

OPTICAL PULSE AND PULSE-TRAIN PROPAGATION IN A RESONANT MEDIUM*

Joseph H. Eberly
Department of Physics and Astronomy, University of Rochester, Rochester, New York 14627
(Received 19 February 1969)

We derive analytic pulse and pulse-train solutions to the nonlinear coupled Maxwell-Schrödinger equations describing the propagation of shape-preserving ultrashort optical pulses in a resonant medium.

A great deal of attention has been given recently to the generation of ultrashort optical pulses[1] and their interaction with atomic systems. The special cases in which the atomic system is either an attenuator[2] or an amplifier[3] (resonant atoms asymptotically in the ground state or excited state, respectively) have been treated in some detail, principally by computer solution of the coupled Maxwell-Schrödinger equations in the quasimonochromatic-field approximation.

We describe here a method which permits analytic solution of this resonant energy-transport problem, regardless of the asymptotic preparation of the medium. The atomic system is assumed to consist of two-level atoms continuously distributed in a nondispersive host medium. The resonant interaction is between the atomic dipole moments and a quasimonochromatic plane-wave electric field. The field center frequency is equal to the center frequency of the symmetric, inhomogeneously broadened atomic line. The relation between atomic polarization and electric field is found to be highly nonlinear, as expected. In addition, the experimental breakup of large pulses into smaller ones is predictable.

The electric field is assumed to be a circularly polarized plane wave modulated by a slowly varying envelope which is a function of the single parameter $\zeta = t - z/v$, $\vec{E}(z,t) = \mathcal{E}(\zeta)[\hat{\imath}\cos\beta + \hat{\jmath}\sin\beta]$ where $\beta = \omega(t - \eta z/c)$, and η is the refractive index of whatever host medium contains the resonant, interacting, two-level, atomic dipoles. Then, in the usual rotating reference frame,[4] the coupled and nonlinear Maxwell and Schrödinger equations for a particular interacting atom may be written in terms of the components (u, v, w) of a unit vector $\vec{r}$. We look for shape-preserving pulse solutions by assuming that u, v, and w also depend only on ζ. Then after writing all relevant space and time derivatives in terms of ζ derivatives we have[5]

$$\dot{u} = -\gamma v, \tag{1}$$

$$\dot{v} = \gamma u + \kappa \mathcal{E} w, \tag{2}$$

$$\dot{w} = -\kappa \mathcal{E} v, \tag{3}$$

$$\dot{\mathcal{E}} = \frac{-2\pi\kappa}{\eta c \Delta} \mathfrak{N} \frac{\hbar\omega}{2} \int v(\zeta, \gamma) g(\gamma) d\gamma. \tag{4}$$

The physical interpretation of u, v, and w is the standard one for $\Delta m = \pm 1$ electric dipole transitions: u and $-v$ are the dimensionless dispersion and absorption components of the dipole moments, and w is the atom energy divided by $\frac{1}{2}\hbar\omega$. $\mathfrak{N}$ is the number of interacting atoms per unit volume, γ is the amount by which the individual atom is off resonance with the field center frequency, $g(\gamma)$ is the atomic line-shape function,

$\frac{1}{2}\hbar\kappa$ is the transition dipole matrix element, and $\Delta = 1/v - \eta/c$.

At this point it is useful to observe the following simple theorem: If $\mathcal{E}$, u, v, w are a particular set of functions satisfying Eqs. (1)-(4), then $\mathcal{E}$, $-u$, $-v$, $-w$ also satisfy Eqs. (1)-(4), but with the sign of Δ reversed. The proof of the theorem is obvious from an inspection of the equations.[6] The theorem suggests the separation of solutions into "slow" and "fast" classes, corresponding to $\Delta > 0$ and $\Delta < 0$, respectively. Thus "slow" solutions travel in the forward direction only, with $c/\eta > v > 0$. All other pulse velocities, both forward and backward, lead to $\Delta < 0$ and thus give rise to "fast" solutions. It is sufficient to consider here slow pulses.

We proceed with the solutions by making the Ansatz[7] $v(\zeta, \gamma) = y(\zeta)\psi(\gamma)$, with $\psi(0) = 1$; furthermore we abbreviate

$$\frac{2\pi}{\eta c \Delta} \mathfrak{N} \frac{\hbar\omega}{2} \int g(\gamma')\psi(\gamma')d\gamma' = \frac{1}{\mu^2} \tag{5}$$

so that Eq. (4) may be written

$$\psi\mu^2\dot{\mathcal{E}} = -\kappa v. \tag{6}$$

Then three first integrals are easily found for Eqs. (1)-(3), (6):

$$u^2 + v^2 + w^2 = 1, \tag{7}$$

$$\tfrac{1}{2}\psi\mu^2\mathcal{E}^2 = w - w_0, \tag{8}$$

$$\gamma\psi\mu^2\mathcal{E} = -\kappa(u - u_0), \tag{9}$$

where u_0 and w_0 are the values of u and w when $\mathcal{E} = 0$.

The coupled equations may be reduced to quadratures by squaring Eq. (3) and using Eqs. (7)-(9) repeatedly to reduce the right-hand side of the squared equation to a function of w alone.[8] In the case $u_0 = 0$, the general solution for arbitrary initial field and inversion may be written explicitly. In that case one has the equation

$$\dot{w}^2 = (2\gamma^2/r)(w - w_0)(1 + 2rw_0 - 2rw - w^2), \tag{10}$$

where

$$r = (\mu\gamma/\kappa)^2\psi,$$

and the integration leads to elliptic functions except in the two cases $w_0 = \pm 1$.

The general solution, for $|w_0| \leq 1$, is

$$w(\zeta) = a^+ - (a^+ - w_0)\,\mathrm{sn}^2[(1/\tau)(\zeta - \zeta^+); k], \tag{11}$$

where $a^\pm = -r \pm (1 + 2rw_0 + r^2)^{1/2}$ are the two roots of the quadratic factor in (10), and the elliptic modulus is given by $k^2 = (a^+ - w_0)/(a^+ - a^-)$. The pulse width is found to be $(\gamma\tau)^2 = (a^- + a^+)/(a^- - a^+)$.

Thus the pulse solution is in general an infinite periodic pulse train and the atom energy oscillates between a^+ and w_0 with period equal to $2K(k)$, where K is the complete elliptic integral of the first kind. The solution for the envelope function $\mathcal{E}(\zeta)$ which accompanies the solution (11) for $w(\zeta)$ is also periodic and is found from Eq. (8):

$$\mathcal{E} = \mathcal{E}_0\,\mathrm{cn}[(\kappa\mathcal{E}_0/2k)(\zeta - \zeta^+); k], \tag{12}$$

where $\mathcal{E}_0$ may also be expressed in terms of τ: $\mathcal{E}_0 = 2k/\kappa\tau$. The physical significance of the modulus k is found by integrating $\mathcal{E}$ over one of its peaks to determine the net turning angle of an atomic dipole. One finds $k = \sin(\frac{1}{4}\kappa\int\mathcal{E}d\zeta)$. Finally, v may be determined to be

$$v = 2k\psi\,\mathrm{cn}[(1/\tau)(\zeta - \zeta^+)]\,\mathrm{dn}[(1/\tau)(\zeta - \zeta^+)], \tag{13}$$

where

$$\psi = (\kappa/\mu\gamma a)^2[w_0 + (w_0^2 + a^2)^{1/2}],$$

and $a^2 = (1/\gamma\tau)^4 - 1$. Clearly ψ can vary greatly from the Lorentzian shape found previously.[2] The consistency condition $\psi(0) = 1$ gives $\mu = \kappa\tau$, from which the pulse velocity can be expressed in terms of τ also.

All of the results are readily reduced to the known[2] nonoscillatory single-pulse solution in the limit $w_0 = -1$, corresponding to the atoms asymptotically in the ground state and no field present. In that case we have $a^+ = 1 - 2r$, $a^- = w_0 = -1$, and $k^2 = 1$. $K(1) = \infty$, so the pulse period is infinite; $\mathrm{sn}[u; 1] = \tanh u$ and $\mathrm{cn}[u; 1] = \mathrm{sech} u$. The value of ψ reduces to $[1 + (\gamma\tau)^2]^{-1}$, and the solutions for u, v, w, $\mathcal{E}$ become those of McCall and Hahn.

In connection with the pulse train solutions derived here, several remarks should be made. In the first place they seem intrinsically interesting in their own right as a class of solutions to a nonlinear resonant energy-transfer problem. The different behavior of these solutions, compared with the known nonoscillatory solutions, is obvious. It is not unreasonable to expect that corresponding relatively stable multipulse, if not actually infinite-pulse, trains can be produced experimentally.

It is important that the solutions have been able to be derived rather than simply verified. Thus the uniqueness conjecture of McCall and Hahn[7] concerning the hyperbolic secant is confirmed. There are no other single-pulse solutions. Furthermore, we have established that there are no true $2m\pi$ pulse solutions for any values of m except 1 and ∞. A large pulse is unstable against evolution toward a long chain of 2π pulses. This would seem to be the origin of the large-pulse breakup observed in computer solutions and discussed by Lamb.[9]

We have not discussed the very interesting "fast" solutions; and can only mention without proof that, for a wide range of initial inversions, pulse propagation seems possible at both of two distinct velocities. In addition, there are solutions for which the field envelope oscillates but is never zero. One of these appears to have been found by Arecchi, DeGiorgio, and Someda.[8] Initial conditions appropriate to a photon echo pulse, with $v_0 \neq 0$, may also be investigated. A detailed discussion of these and other related topics is in preparation.

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Note added in proof.—While this paper was in preparation, E. T. Jaynes called our attention to a similar investigation being undertaken by M. D. Crisp.

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[5]The equations are the same as those referred to by McCall and Hahn (Ref. 2) except for insignificant sign changes. We have also assumed that phenomenological relaxation times are longer than the pulse duration and may be ignored.

[6]It is interesting to observe that there is a 2π pulse solution to the amplifier equations of Arecchi and Bonifacio which is conjugate in the sense of the theorem to the McCall-Hahn solution. We are happy to thank several persons for pointing out that this amplifier solution is new, and does not coincide with one of the Arecchi-Bonifacio steady-state pulse solutions.

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ANOMALOUS SKIN EFFECT IN BOUNDED PLASMAS

J. A. Reynolds, H. A. Blevin,* and P. C. Thonemann†
United Kingdom Atomic Energy Authority Research Group, Culham Laboratory, Abingdon, Berkshire, England
(Received 17 February 1969)

The penetration of an electromagnetic field into a cylindrical plasma has been measured under conditions where the electron mean free path is comparable with the plasma diameter. The experimental results are in qualitative agreement with a theoretical model in which electron thermal motion is included.

An electromagnetic wave incident on a bounded plasma with no static magnetic field present is strongly attenuated when $\omega \ll \omega_p$ (ω is the wave frequency, ω_p the plasma frequency). The skin depth is usually calculated using the complex conductivity $\sigma = Ne^2/m(i\omega + \nu)$, where N, e, m, and ν are the electron density, charge, mass, and collision frequency, respectively. However, when electrons travel a distance comparable with the skin depth during a wave period (due to their thermal motion), a local relation between the electric field and the current density no longer provides a valid description of wave attenuation.[1-3] In this communication we describe an experiment which demonstrates the importance of electron thermal motion and compare the results with a simplified theory.

The plasma was produced by a pulsed discharge through low-pressure mercury vapor in a cylindrical glass tube 150 cm long and 8.1 cm diam. After a 20-μsec current pulse of 250 A, a steady current of 13.7 A was passed through the plasma, maintaining the electron temperature at about 2 eV. The electron-neutral collision frequency ν_{en}