SPECTRUM AND ASYMMETRY OF DIRECT NEUTRONS FROM MUON CAPTURE IN CALCIUM*

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The energy spectrum and asymmetry of neutrons emitted directly on the capture of polarized muons by ^{40}Ca have been calculated in a simple nuclear shell model, taking account of terms proportional to the nucleon momentum. The energy spectrum is within the error brackets of recent experiments with the nucleon-momentum terms making a small contribution. The asymmetry coefficient is strongly affected by the inclusion of the momentum-dependent terms and is found to change sign and increase in magnitude with increasing energy, in qualitative agreement with recent experiments.

In the strangeness-conserving leptonic weak interaction

$$\mu^- + {}_Z^AX_N \to {}_{Z-1}^{A-1}X_N + n + \nu_\mu, \tag{1}$$

the number of neutrons emitted per muon capture per unit energy per steradian is of the form

$$N_c(E,\Theta) = \frac{N_{ct}(E)}{4\pi}[1 + P\alpha(E)\cos\Theta], \tag{2}$$

where P is the muon polarization, Θ is the angle between the (asymptotic) neutron direction and the muon polarization, and where $\alpha(E)$ and the total number of neutrons per capture per unit energy $N_{ct}(E)$ depend upon the outgoing neutron energy E. Both $\alpha(E)$ and $N_{ct}(E)$ have been the object of experimental[1-3] and theoretical[4-13] interest.

In this Letter, we report a calculation of $N_{ct}(E)$ and $\alpha(E)$ for neutrons emitted directly (without compound-nucleus formation) which agrees with the latest experimental values of $N_{ct}(E)$ at high energies[1] (where direct processes dominate) and gives positive values of α for high energies, in qualitative agreement with recent experiments.[1,2]

Our calculation is for ^{40}Ca but may easily be extended to other spin-zero nuclei. Our model will be discussed in detail in a subsequent paper[14] and is only sketched here:

(1) The effective Hamiltonian for muon capture of Fujii and Primakoff[15] is used and terms proportional to the nucleon momenta $\hbar(\vec{p}+\vec{n})/m_pc$ (where $\hbar\vec{a}$ is the momentum of particle a) are brought to the form $\hbar\vec{\nu}/m_pc + 2\hbar\vec{n}/m_pc$ by momentum conservation $\vec{n}+\vec{\nu}\approx\vec{p}$, where the approximation results from neglecting the muon momentum.[16]

(2) The initial nuclear wave function is assumed to consist of a sum of antisymmetrized products of core and single-proton harmonic-oscillator states and the final state consists of the core and a plane-wave neutron.[13]

(3) The dynamics of the final-state interaction between the neutron and the residual nucleus are ignored, and the neutron is considered to pass with its direction unchanged through the nuclear surface.

(4) Part of the interaction between the neutron and the residual nucleus is included by using, for the neutron momentum inside the nuclear volume,

$$(\hbar n)^2 = 2m^*m_p(E+B_w) + P_F^{\ 2}, \tag{3}$$

where m_p is the mass of the proton, B_w is the binding energy of the most weakly bound nuclear proton, m^* is the neutron's effective-mass ratio inside the nucleus, and $P_F = 268.0$ MeV/c is the nuclear Fermi momentum.[17]

The final nuclear wave function is an eigenstate of $\vec{n}$ when plane waves are taken for the neutron wave function, and the nuclear matrix elements involving $\hbar\vec{n}/m_pc$ may be expressed as products of $\hbar\vec{n}/m_pc$ and simpler matrix elements. The square of the matrix element for muon capture $\mathfrak{M}$ then becomes (when summed over final lepton spins, averaged over the initial lepton spins, and integrated over the neutrino direction)

$$\int d\vec{\Omega}_\nu \sum_f\sum_i |\mathfrak{M}|^2 = 2\pi(2J+1)G^2\langle|\varphi_\mu|^2\rangle\Big[\mathcal{G}_0T_0(\nu,n) + \frac{\hbar n}{m_pc}\mathcal{G}_2T_1(\nu,n) + \cos\Theta\Big\{\mathcal{G}_1T_1(\nu,n) + \frac{\hbar n}{m_pc}\mathcal{G}_3T_0(\nu,n) + \frac{\hbar n}{m_pc}\mathcal{G}_4T_2(\nu,n)\Big\}\Big], \tag{4}$$

where the muon wave function φ_μ has been assumed constant over the nuclear volume and brought out of the nuclear matrix element,[10,18] and where $G = 1.01\times 10^{-5}\hbar^3/m_p^2c$ is the basic weak-interaction coupling constant.[19] The coupling constant ratios are defined by

$$\mathcal{G}_0 \equiv (G_V'^2 + 3G_A^2 + G_P'^2 - 2G_A G_P')/G^2, \tag{5a}$$

$$\mathcal{G}_1 \equiv (G_A^2 - G_V'^2 - G_P'^2 + 2G_A G_P')/G^2, \tag{5b}$$

$$\mathcal{G}_2 \equiv 2[(G_P' - G_A)g_A - G_V' g_V]/G^2, \tag{5c}$$

$$\mathcal{G}_3 \equiv 2(G_V' g_V + G_A g_A)/G^2, \tag{5d}$$

$$\mathcal{G}_4 \equiv -2G_P' g_A/G^2, \tag{5e}$$

where the effective coupling constants are

$$G_V' \equiv g_V\left(1 - \frac{\hbar\nu}{2m_p c}\right) = G_V - \frac{\hbar\nu}{m_p c} g_V, \tag{6a}$$

$$G_P' \equiv [(g_P + g_A) - (g_V + g_M)]\frac{\hbar\nu}{2m_p c} = G_P + g_A \frac{\hbar\nu}{m_p c} \tag{6b}$$

in terms of the usual[10,18] apparent vector, axial-vector, induced pseudoscalar, and weak-magnetism coupling constants g_V, g_A, g_P, and g_M. The nuclear matrix elements $T_L(\nu, n)$ are given by

$$T_L(\nu, n) = \int d\vec{\Omega}_\nu (\hat{\nu}\cdot\hat{n})^L \left|\int r^2 dr\, j_l(|\vec{\nu}+\vec{n}|r) R_{NJl}(r)\right|^2, \tag{7}$$

where $R_{NJl}(r)$ is the radial part of the wave function for protons with energy, total angular momentum, and orbital angular-momentum quantum numbers N, J, and l, where $j_l(|\vec{\nu}+\vec{n}|r)$ is the spherical Bessel function of order l and where $\hat{a}$ is a unit vector in the direction of $\vec{a}$. The nuclear matrix elements take the simple form (7) again because we have assumed plane-wave neutron states, allowing us to expand the final plane-wave neutron and neutrino wave function in spherical harmonics together and to use the orthonormality of the spherical harmonics to eliminate all terms except the term of order l, the proton orbital angular momentum. This is the procedure followed by Devanathan and Rose,[12] who give closed-form expressions for T_L for all the proton shells in ^{40}Ca, which may be written

$$T_L(\nu, n)/[4\pi^{\frac{3}{2}} b^3 e^{-b^2(n^2+\nu^2)}] \equiv T_L'(\nu, n)$$

$$= 2f_L(z) \quad \text{for } N=1, l=0; \tag{8a}$$

$$= \tfrac{4}{3}[b^2(n^2+\nu^2) + L + 1]f_L(z) + za_{L+1}(z) \quad \text{for } N=1, l=1; \tag{8b}$$

$$= (8/15)\{[b^4(n^2+\nu^2)^2 + (L+1)2b^2(n^2+\nu^2) + (L+2)(L+1)]f_L(z)$$

$$+ [2b^2(n^2+\nu^2) + L + 2]za_{L+1} + z^2 a_L(z)\} \quad \text{for } N=1, l=2; \tag{8c}$$

and

$$T_L'(\nu, n)|_{N=2, l=0} = \tfrac{3}{2}T_L'(\nu, n)|_{N=1, l=0} - 3T_L'(\nu, n)|_{N=1, l=1} + \tfrac{5}{2}T_L'(\nu, n)|_{N=1, l=2}. \tag{8d}$$

Here b is the usual harmonic-oscillator length parameter,[10]

$$z \equiv 2b^2 n\nu, \tag{9}$$

$$a_L(z) = z^{-1}\sinh z = f_0(z) \quad \text{for } L \text{ even}, \tag{10a}$$

$$a_L(z) = -z^{-1}\cosh z \quad \text{for } L \text{ odd,} \tag{10b}$$

and

$$f_L(z) = a_L(z) + Lf_{L-1}(z)/z \quad \text{for } L \geq 1. \tag{11}$$

The differential rate for capture by a given proton shell is given by perturbation theory:

$$\begin{aligned}\frac{d\Lambda}{dE_n d\vec{\Omega}_n} &= \frac{2\pi}{\hbar}\left(\frac{d^3\vec{n}_0}{dE_n d\vec{\Omega}_n}\right)\frac{1}{(2\pi)^6}\int d^3\nu\,\delta(m_\mu c^2 - B_p - E - \hbar\nu c)\sum_f\overline{\sum_i}|\mathfrak{M}|^2 \\ &= \frac{1}{4\pi^{7/2}}\frac{m_p Z_{\text{eff}}^4 G^2}{\hbar^4 c a_0^3 Z}(b\nu)^2(bn_0)e^{-b^2(n^2+\nu^2)}(2J+1) \\ &\times\left[\mathcal{G}_0 T_0'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_2 T_1'(\nu,n) + \cos\Theta\left\{\mathcal{G}_1 T_1'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_3 T_0'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_4 T_2'(\nu,n)\right\}\right],\end{aligned} \tag{12}$$

where we have used the relation[20] $Z_{\text{eff}}^4 = \pi a_0^3 Z\langle|\varphi_\mu|^2\rangle$, $a_0 = \hbar^2/m_\mu e^2$ is the muon Bohr radius, B_p is the binding energy of protons in this shell, and where we have used the free-space momentum $\hbar n_0 = (2m_p E)^{1/2}$ in the neutron phase-space factor.

The quantity $N_c(E,\Theta)$ is given by $N_c(E,\Theta) = (d\Lambda/dE_n d\vec{\Omega}_n)(1/\Lambda_T)$, where Λ_T is the total capture rate. Thus, from Eq. (2),

$$N_{ct}(E) = \frac{1}{149.3}(b\nu)^2(bn_0)e^{-b^2(n^2+\nu^2)}(2J+1)\left[\mathcal{G}_0 T_0'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_2 T_1'(\nu,n)\right](\text{MeV})^{-1} \tag{13}$$

and

$$\alpha(E) = \left[\mathcal{G}_1 T_1'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_3 T_0'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_4 T_2'(\nu,n)\right]\left[\mathcal{G}_0 T_0'(\nu,n) + \frac{\hbar n}{m_p c}\mathcal{G}_2 T_1'(\nu,n)\right]^{-1} \tag{14}$$

for capture from one shell, where we have evaluated the constants in Eq. (12) and taken[21,22] $\Lambda_T/Z_{\text{eff}}^4 = 36.573\ \text{sec}^{-1}$. The quantity $N_{ct}(E)$ for the entire nucleus is obtained by summing the contributions from each shell (13), but for the asymmetry coefficient $\alpha(E)$ one must sum the coefficients of $\cos(\Theta)$ in Eq. (12) and divide by the remaining terms.

The values of the remaining quantities needed to calculate $N_{ct}(E)$ and $\alpha(E)$ are given in Table I. The proton binding energies have been estimated from experiments[23-25] on the reaction ^{40}Ca(p, $2p$)^{39}K. We have chosen the usual values for the apparent coupling constants,[15,18] b is chosen[10] to give the experimental rms charge radius, and m^* is taken from the Brueckner-Goldstone theory of nuclear matter.[17] Our results for $N_{ct}(E)$ and $\alpha(E)$ are also given in Table I for values of E from 0.0 to 100.0 MeV and are plotted in Fig. 1 along with the experimental values for $N_{ct}(E)$ of Sundelin, Edelstein, Suzuki, and Takahashi[1] and Sundelin[26] for ^{40}Ca. The agreement between our theory and experiment for $N_{ct}(E)$ at high energies, where direct processes dominate, is remarkable in view of the large discrepancy between the experimental values and values from older theories.[6-8]

In the graph of the asymmetry parameter in Fig. 1, the agreement between our theory and experiment is greater than is at first apparent. First, our theory appears to be the first to predict positive asymmetry coefficients for high energies.[27] Second, the energy E for the experimental values does not represent the energy of the neutron detected, but rather the lowest energy of the neutron which can give sufficient energy to the proton to be detected in a given energy bin. Consequently, the experimental value of $\alpha(E)$ plotted includes contributions from more energetic neutrons. The statistics of the experiment did not permit unfolding of those data for $\alpha(E)$ as they did for $N_{ct}(E)$. Preliminary results of a statistically improved version of this experiment[28]

Table I. Preferred values of physical parameters and results for the asymmetry coefficient and the number of neutrons per capture per energy.

Parameter	Value		E(MeV)	$\hbar n/m_p c$ terms included α	$N_{ct}(\mathrm{MeV})^{-1}$	$\hbar_n/m_p c$ terms omitted α	$N_{ct}(\mathrm{MeV})^{-1}$
$B(1d^{3/2})$	8.5	MeV[a,b,c]	1	-0.963	4.96×10^{-3}	-0.284	5.57×10^{-3}
$B(2s)$	11.1	MeV[a,b,c]	5	-0.080	7.50×10^{-3}	-0.277	8.41×10^{-3}
$B(1d^{5/2})$	14.5	MeV[a,b,c]	10	-0.059	6.25×10^{-3}	-0.267	7.00×10^{-3}
$B(1p^{1/2})$	17.0	MeV[a]	15	-0.037	4.33×10^{-3}	-0.256	4.84×10^{-3}
$B(1p^{3/2})$	20.0	MeV[b]	25	0.010	1.58×10^{-3}	-0.231	1.76×10^{-3}
$B(1s)$	60.0	MeV[c]	35	0.060	4.49×10^{-4}	-0.202	5.04×10^{-4}
g_V	0.983	G[d]	45	0.112	1.04×10^{-4}	-0.171	1.18×10^{-4}
g_A	- 1.15	G[d]	55	0.167	2.00×10^{-5}	-0.137	2.33×10^{-5}
g_P	7.5	g_A[d]	65	0.225	3.07×10^{-6}	-0.100	3.76×10^{-6}
g_M	3.69	G[d]	75	0.289	3.45×10^{-7}	-0.062	4.74×10^{-7}
b	2.03	F[e]	85	0.356	1.89×10^{-8}	-0.027	3.60×10^{-8}
m^*	0.6[f]		95	0.425	7.11×10^{-12}	-0.001	2.66×10^{-11}

[a]Ref. 23.
[b]Ref. 24.
[c]Ref. 25.
[d]Refs. 15 and 18.
[e]Ref. 10.
[f]Ref. 17.

may permit such unfolding and also seem to reduce the values of $N_{ct}(E)$ for the two largest energy bins, bringing them into closer agreement with our theory.

We have also included in Table I the values for $N_{ct}(E)$ and $\alpha(E)$ when the terms in the effective Hamiltonian proportional to the nucleon momentum are omitted. These quantities may be calculated from Eq. (12) by setting $\mathcal{G}_2 = \mathcal{G}_3 = \mathcal{G}_4 = 0$ and using G_P and G_V rather than G_P' and G_V' in the definitions of $\mathcal{G}_0$ and $\mathcal{G}_1$.[29] These values of $\alpha(E)$ are similar to those of others—they are negative and decrease in magnitude as E increases. The effect of the additional terms on $N_{ct}(E)$ is not very great, however. As we will see, our values for $N_{ct}(E)$ are improved principally because of our use of an effective mass different from unity.

The variation of α and N_{ct} with energy, effective mass, and oscillator parameters for the various shells is exhibited in Table II. Our result for the asymmetry coefficient α varies from shell to shell and large values for α are usually associated with small neutrino energy (large neutron energy or proton-binding energy). The asymmetry coefficient decreases slightly with increasing oscillator parameter and usually in-

Table II. The asymmetry coefficient and the number of neutrons per capture per unit energy for the various shells and some values of the effective mass and oscillator parameters.

E(MeV)	Shell	b = 2.03 F, m^* = 0.6 α	$N_{ct}(\mathrm{MeV})^{-1}$	b = 2.03 F, m^* = 1.0 α	$N_{ct}(\mathrm{MeV})^{-1}$	b = 2.08 F, m^* = 0.6 α	$N_{ct}(\mathrm{MeV})^{-1}$
5	All	-0.0804	7.50×10^{-3}	-0.0795	4.29×10^{-3}	-0.0880	6.62×10^{-3}
25	All	0.0099	1.58×10^{-3}	0.0201	2.69×10^{-4}	0.0037	1.22×10^{-3}
45	All	0.1120	1.04×10^{-4}	0.1305	4.00×10^{-6}	0.1064	6.93×10^{-5}
45	1s	—	0.0	—	0.0	—	0.0
45	$1p^{3/2}$	0.1449	3.00×10^{-6}	0.1670	7.84×10^{-8}	0.1401	1.85×10^{-6}
45	$1p^{1/2}$	0.1296	1.96×10^{-6}	0.1518	5.30×10^{-8}	0.1249	1.22×10^{-6}
45	$1d^{5/2}$	0.1255	3.44×10^{-5}	0.1450	1.26×10^{-6}	0.1201	2.26×10^{-5}
45	2s	0.1114	2.84×10^{-5}	0.1296	1.16×10^{-6}	0.1057	1.91×10^{-5}
45	$1d^{3/2}$	0.0962	3.66×10^{-5}	0.1157	1.44×10^{-6}	0.0909	2.45×10^{-5}

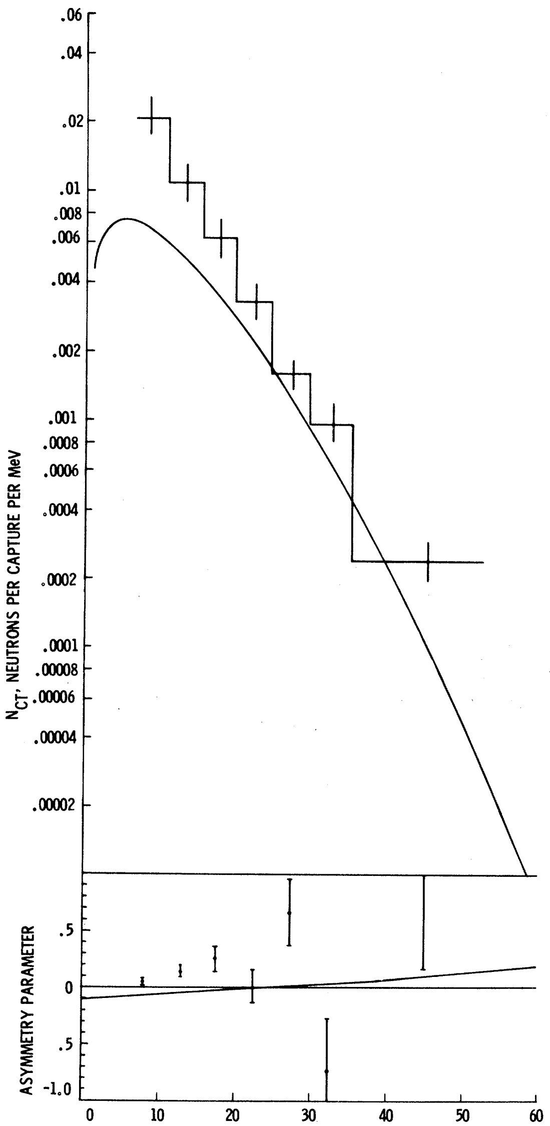

FIG. 1. Comparison of theoretical predictions and experimental measurements (see Ref. 1) of the neutron spectrum and asymmetry parameters.

creases slightly with increasing effective mass. The difference between our results for the asymmetry parameter and those of others is clearly the result of additional terms in the effective Hamiltonian. In the calculation of an asymmetry parameter, which involves the difference of matrix elements, cancellation of the major terms may allow the momentum-dependent terms (which are usually less than 10% of the others) to assume a major role.

Although the neutron spectrum is not strongly affected by the momentum-dependent terms, we see in Table II that the neutron intensity N_{ct} does depend strongly on the choice of parameters. The neutron intensity is reduced either by increasing the oscillator parameter or by increasing the effective mass. Both these effects may be explained by considering the factor $e^{-b^2(n^2+\nu^2)}$ in Eq. (13) since the neutron wave number n will increase with the effective mass. This factor also explains the decrease of intensity with neutron energy. The neutrino momentum is always less than $m_\mu c$ (= 105.66 MeV/c), while the neutron momentum is always greater than the neutron Fermi momentum (= 268.0 MeV/c); so the effect of an increase of ν in the exponential factor will be masked by n, whereas the factor $(b\nu)^2$ in Eq. (13) will increase N_{ct}. We see the effect of an increase of ν (or a reduction of proton-binding energy) clearly in Table II by the variation of N_{ct} among the shells. Deviations from this rule result from the differing number of capturing protons in a shell $(2J+1)$.

In summary, the asymmetry parameter is greatly affected by the inclusion of momentum-dependent terms in the effective Hamiltonian. These terms shift the values of α toward more positive values and are more pronounced at high exit neutron energies. The neutron intensity, on the other hand, is sensitive to the details of the variation of the neutron wave function. Our assumptions concerning the neutron wave function differ greatly from the assumptions made for direct nuclear processes because there the reaction takes place primarily at the nuclear surface, while here the neutron is produced in the nuclear interior. The close agreement between our results for the neutron intensity and experiment may be fortuitous in view of the sensitivity of the intensity upon the parameters, but this same sensitivity makes the process an excellent probe of nuclear properties.

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[29]By using G_V and G_P, we neglect terms of order $\hbar\vec{p}/m_pc$ rather than $\hbar\vec{n}/m_pc$ (see Refs. 4 and 11).