RELATION BETWEEN THE AVERAGE REACTION CROSS SECTION AND THE TRANSMISSION COEFFICIENTS*

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An expression is derived for the average reaction cross section using the two-channel, multilevel form of the unitary low-energy scattering matrix. It expresses the average cross section as a function of the transmission coefficients only and is valid for all values of the transmission coefficients. It is shown that over a wide range of the values of the transmission coefficients the values calculated using the Hauser-Feshbach expression agree very well with the present model.

It has been known[1,2] for quite some time that the Hauser-Feshbach expression[3,4] for the average reaction cross section, which is derived under the assumption that the transmission coefficients T_c for the various channels c are much less than unity, fits the data nicely even if the values of the T_c's are not much less than unity. In the recent statistical analysis of the heavy-ion reaction data by Gadioli et al.[2] for the cases where the transmission coefficients were definitely not much less than unity, they found that if they employ the improved relation[5,6] between the transmission coefficient and the ratio of the average partial width to the average spacing, then their calculation overestimates the cross sections. This procedure would have been correct had the authors used a better expression connecting the average reaction cross section with the ratios of the average partial widths to the average spacing. The two main difficulties in deriving such an expression are (1) keeping track of the unitarity constraint on the low-energy scattering matrix and (2) avoiding the approximations which are usually valid when the ratio of the average partial width to the average spacing is much less than unity. Only recently[5,6] has it been possible to overcome these difficulties partially and derive expressions for the transmission coefficients, which are valid for both the sharp as well as overlapping resonances. The object of the present work is to develop the earlier formalism[6] further and derive an expression for the average cross section in terms of the transmission coefficients for the case of two channels. This expression will then be compared with a similar expression which is derived using the picket-fence model[5] and also with the Hauser-Feshbach expression.[3,4] The comparison between the present expression and the Hauser-Feshbach expression will show why the latter remains a good approximation even when the T_c's are not much less than unity.

We now give the derivation of the expression for the average cross section. The low-energy scattering matrix S is written in the form[7]

$$S(E)=V\left[1-i\sum_{\mu=1}^{N}\frac{G_\mu\times G_\mu}{E-z_\mu}\right]V, \tag{1}$$

where the background matrix V is the unitary and z_μ's are the complex poles of S, $z_\mu=\epsilon_\mu-\frac{1}{2}i\Gamma_\mu$. The complex quantities G_μ are the vectors in the channel space with components $G_{\mu c}$. Let us now consider the case of two channels c, c'; then unitarity implies the following relations[7] between G_μ's and z_μ's:

$$\sum_{\mu=1}^{N}B_{\nu\mu}G_{\mu c'}^{*2}=g_\nu-G_{\nu c}^{2}, \tag{2a}$$

$$\sum_\mu B_{\nu\mu}G_{\mu c}^{*2}=g_\nu-G_{\nu c'}^{2}, \tag{2b}$$

$$\sum_\mu B_{\nu\mu}G_{\mu c}^{*}G_{\mu c'}^{*}=G_{\nu c}G_{\nu c'}, \tag{2c}$$

where g_ν and $B_{\nu\mu}$ are given by

$$g_\nu=i[\prod_\lambda(z_\nu-z_\lambda^*)][\prod_{\lambda\neq\nu}(z_\nu-z_\lambda)]^{-1}, \tag{3a}$$

$$B_{\nu\mu}=ig_\nu/(z_\mu^*-z_\nu). \tag{3b}$$

The average reaction cross section $\langle\sigma_{cc'}\rangle$ is defined by

$$\langle\sigma_{cc'}\rangle=(\pi/k_c^2)\langle|S_{cc'}(E)|^2\rangle, \tag{4}$$

where $\langle\ \rangle$ denotes an energy average[8] and k_c is the wave number in channel c. Even though it is possible to derive an expression for $\langle|S_{cc'}(E)|^2\rangle$ with V a general unitary and symmetric matrix, we shall not do this here, since our main interest is in the resonance part of $S(E)$. We therefore put the matrix $V=1$.

The energy average of $|S_{cc'}(E)|^2$ using the relations (2) is given by[7]

$$\langle |S_{cc'}(E)|^2\rangle = 2\,\mathrm{Re}\left[i\sum_{\mu} \frac{G_{\mu c}{}^2 G_{\mu c'}{}^2/g_\mu}{E_0+\frac{1}{2}iI-z_\mu}\right]. \qquad (5)$$

It is easy to show using the relations (2) that the quantity $\sum_\mu G_{\mu c}{}^2 G_{\mu c'}{}^2/g_\mu$ is real. Using an earlier technique[6] to evaluate the sum in expression (5) when both I and N become large, such that the ratio I/N remains constant, we find that $\langle |S_{cc'}(E)|^2\rangle$ is given by

$$\langle |S_{cc'}(E)|^2\rangle = \left[1-\exp\left(-\frac{2\pi}{D}\left\langle \frac{G_{\mu c}{}^2 G_{\mu c'}{}^2}{g_\mu}\right\rangle\right)\right], \qquad (6)$$

where D is the average spacing.

The next step in our derivation is to eliminate the real quantities $\langle G_{\mu c}{}^2\rangle$, using the expressions which connect them with the transmission coefficients.[6] Without the width fluctuation factor the Hauser-Feshbach expression is written as

$$\langle \sigma_{cc'}{}^{\mathrm{H.F.}}\rangle = \frac{\pi}{k_c{}^2}\frac{T_c T_{c'}}{T_c+T_{c'}}. \qquad (7)$$

To arrive at the corresponding expression in our two-channel model, we take the probability distribution of $G_{\mu c}$ independent of $G_{\mu c'}$. This together with the expression for the transmission coefficients[6] and expressions (4) and (6) immediately give us the desired expression

$$\langle \sigma_{cc'}\rangle = \frac{\pi}{k_c{}^2}\left[1-\exp\left(\frac{[\ln(1-T_c)][\ln(1-T_{c'})]}{[\ln(1-T_c)+\ln(1-T_{c'})]}\right)\right]. \qquad (8)$$

Before we discuss the results we also give the expression for the average reaction cross section using the picket-fence model.[5] It is given by

$$\langle \sigma_{cc'}{}^{\mathrm{p.f.}}\rangle = \frac{\pi}{k_c{}^2}\frac{T_c T_{c'}}{T_c+T_{c'}-T_c T_{c'}}. \qquad (9)$$

The first remark which we would like to pass here is on the approximation[9,10] of replacing

$$\sum_\mu \frac{a_\mu}{E_0+\frac{1}{2}iI-z_\mu} \text{ by } -i\frac{\pi}{D}\langle a_\mu\rangle,$$

which is used in a number of theoretical formulations including the expression for the Hauser-Feshbach cross section. Had we used this approximation in expression (5), we would have ended up with the following expression for the average reaction cross section:

$$\langle \sigma_{cc'}{}^{\mathrm{approx}}\rangle = \frac{\pi}{k_c{}^2}\frac{[\ln(1-T_c)][\ln(1-T_{c'})]}{[\ln(1-T_c)]+[\ln(1-T_{c'})]}. \qquad (10)$$

It should be pointed out here that expression (10) is the same which was used by Gadioli et al.[2] in the hope that it will fit the cross sections better for large values of the transmission coefficients, since it can also be obtained by using the improved relation[5,6] between the transmission coefficient and the ratio of the average partial width to the average spacing in the well-known expression [4,10] for the average of $|S_{cc'}(E)|^2$.

For small values of T_c and $T_{c'}$, expression (8) reduces to expression (7) as it should, but when T_c and $T_{c'}$ approach unity, $\langle\sigma_{cc'}{}^{\mathrm{H.F.}}\rangle$ approaches its maximum value of 0.5 in the units of $\pi/k_c{}^2$ while both $\langle\sigma_{cc'}\rangle$ and $\langle\sigma_{cc'}{}^{\mathrm{p.f.}}\rangle$ approach

Table I. Average reaction cross sections based on various models for the case of two channels c, c'. $T_c, T_{c'}$ are the transmission coefficients for the channels c, c'. All cross sections are expressed in the units of $\pi/k_c{}^2$. H.F. = Hauser-Feshbach, p.f. = picket fence, approx = cross section based on the approximation which leads to expression (10), and M = present model.

$T_{c'}$	T_c	H.F.	p.f.	Approx.	M
0.1	0.1	0.0500	0.0526	0.0527	0.0514
	0.3	0.0750	0.0811	0.0814	0.0780
	0.5	0.0833	0.0909	0.0915	0.0874
	0.7	0.0875	0.0959	0.0969	0.0924
	0.9	0.0900	0.0989	0.1008	0.0959
0.5	0.1	0.0833	0.0909	0.0915	0.0874
	0.3	0.1875	0.2308	0.2355	0.2098
	0.5	0.2500	0.3333	0.3466	0.2929
	0.7	0.2917	0.4118	0.4399	0.3559
	0.9	0.3214	0.4737	0.5328	0.4130
0.9	0.1	0.0900	0.0989	0.1008	0.0959
	0.3	0.2250	0.2903	0.3089	0.2657
	0.5	0.3214	0.4737	0.5328	0.4130
	0.7	0.3938	0.6495	0.7906	0.5464
	0.9	0.4500	0.8182	1.1513	0.6838

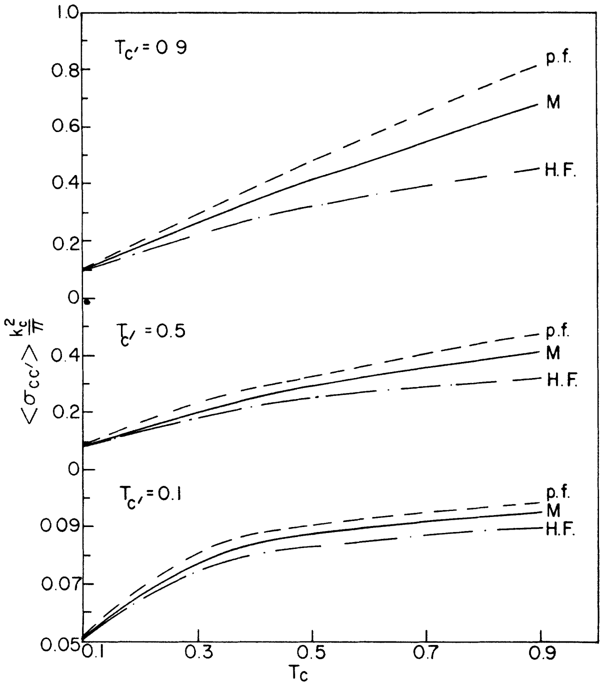

FIG. 1. Plots of the average reaction cross section in the units of π/k_c^2 using the Hauser-Feshbach model (H.F.), picket-fence model (p.f.), and the present model (M) for the various values of the transmission coefficents.

Table II. A comparative study of the Hauser-Feshbach and the present model in the presence of many open channels. H.F. = Hauser-Feshbach cross section using expression (11), M = cross section based on the present model given by expression (12), and $T_{cn} = T_{c''}$, $n = 1, \cdots, 10$.

$T_{c'}$	$T_{c''}$	T_c	H.F.	M
		0.1	0.03333	0.03500
0.1	0.01	0.5	0.07143	0.07800
		0.9	0.08182	0.09220
		0.1	0.03125	0.03867
0.5	0.1	0.5	0.12500	0.17873
		0.9	0.18750	0.32573
		0.1	0.00893	0.00941
0.5	0.5	0.5	0.04167	0.05613
		0.9	0.07031	0.14852
		0.1	0.08182	0.09220
0.9	0.01	0.5	0.30000	0.40275
		0.9	0.42632	0.67587
		0.1	0.00900	0.00950
0.9	0.9	0.5	0.04327	0.05950
		0.9	0.07500	0.17460

unity. This is allowed by the unitarity constraint $\langle |S_{cc}|^2\rangle + \langle |S_{cc'}|^2\rangle = 1$. It should be noted here that the approach of $\langle\sigma_{cc'}\rangle$ to its maximum value unity is much slower than $\langle\sigma_{cc'}{}^{\mathrm{p.f.}}\rangle$.

In Table I we present the calculated values of the average cross sections using the expressions (7)-(10) for the various values of T_c, $T_{c'}$. A comparison of the values calculated using the approximate expression (10) with the ones given by the correct expression (8) shows that the approximate expression overestimates the average cross sections. In search of a better expression for the average cross section when the transmission coefficients are large, Gadioli et al.[2] had tried to use expression (10) to analyze the heavy-ion reaction data and had rightly concluded that it overestimates the cross sections. For values of $T_c \leq 0.5$, the values calculated using the correct expression (8) are not very different than the ones given by the Hauser-Feshbach expression (7). This explains why the Hauser-Feshbach expression fits the data nicely[1,2] even when the T_c's are not much less than unity. In Fig. 1 we have plotted $\langle\sigma_{cc'}{}^{\mathrm{H.F.}}\rangle$, $\langle\sigma_{cc'}\rangle$, and $\langle\sigma_{cc'}{}^{\mathrm{p.f.}}\rangle$ for the various values of T_c, $T_{c'}$. A glance at this figure shows that $\langle\sigma_{cc'}\rangle$ is closer to $\langle\sigma_{cc'}{}^{\mathrm{H.F.}}\rangle$ than $\langle\sigma_{cc'}{}^{\mathrm{p.f.}}\rangle$ for all values of T_c, $T_{c'}$.

It will be interesting to see how expressions (7) and (8) compare when the number of channels is greater than two. For $m+2$ channels we write expression (7) as

$$\langle\sigma_{cc'}{}^{\mathrm{H.F.}}\rangle = \frac{\pi}{k_c^2}\,\frac{T_c T_{c'}}{T_c + T_{c'} + \sum_{n=1}^{m} T_{cn}}, \tag{11}$$

and assume in analogy with expression (8) that $\langle\sigma_{cc'}\rangle$ can be written as

$$\langle\sigma_{cc'}\rangle = \frac{\pi}{k_c^2}\left[1 - \exp\left(\frac{[\ln(1-T_c)][\ln(1-T_{c'})]}{[\ln(1-T_c)] + [\ln(1-T_{c'})] + \sum_{n=1}^{m}[\ln(1-T_{cn})]}\right)\right]. \tag{12}$$

In Table II we show a comparison of the values calculated using the expressions (11) and (12). Again we find that the agreement between the two sets of values is fairly good for a wide range of the values of T_c's.

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PHOTODISINTEGRATION OF THE DEUTERON FROM 220 to 340 MeV*

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The differential cross section for the photodisintegration of the deuteron has been measured at laboratory photon energies of 222, 254, 302, 342 MeV in steps of 10° between 20° and 160° in the center-of-mass system.

Recent speculations[1,2] have made it very desirable to check experimentally CP or T invariance in the electromagnetic interactions of the hadrons. One possible test[3] of T invariance is to compare the reaction

$$\gamma + d \rightarrow n + p$$

with the time-reversed reaction

$$n + p \rightarrow d + \gamma$$

in the region around the first resonance. The reciprocity check in this energy region could be sensitive to a breakdown of time-reversal invariance in the γNN^* vertex. The most sensitive reciprocity test would presumably be a comparison of the angular distributions of the two reactions.

The capture reaction is presently being studied by several groups which are in the process of reducing their data. The photodisintegration differential cross section in this energy region has been measured by several groups.[4-7] The agreement between the two most recent photodisintegration measurements is good at forward angles but there is a definite disagreement at backward angles, and the experiments contain only six to seven points for each angular distribution. We report here measurements in 10° steps between 20° and 160° in the center-of-mass system at four energies from 222 to 340 MeV.

The experiment was carried out using the Stanford Mark III linear accelerator. The momentum-analyzed electron beam ($\Delta E/E = 0.5\%$) was focused upon a copper radiator, 0.05 radiation length thick. After passing through the radiator foil, the electrons were deflected into a beam dump, and the photon beam was collimated to 1-in. diam and passed through the deuterium target.

The target cell was a vertical cylinder of 3 in. diam with nickel-plated stainless-steel walls (0.001-in. stainless steel, 0.0005-in. nickel). The target was of the condensation type and consisted of three such cells. The first was normally filled with liquid deuterium, the second with liquid hydrogen, and the third was empty. The intensity of the photon beam was measured by a secondary-emission quantameter located behind the target. The device was calibrated several independent ways and the reproducibility of the cal-