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STABILITY OF STRAIGHT MULTIPOLES WITH SHEAR*

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(Received 14 November 1968)

Every scalar-pressure, perfectly conducting, two-dimensional equilibrium with three magnetic field components and an arbitrary pressure profile but no current in the ignorable direction is magnetohydrodynamically stable. The stability of these equilibria bears no visible relation to any magnetic well or average magnetic well criterion and is only partly ascribable to shear.

Consider a perfectly conducting, scalar-pressure magnetohydrodynamic equilibrium, subject to $\nabla p = \vec{J}\times\vec{B}$. The second variation of total energy, $\int(\frac{1}{2}B^2+\frac{3}{2}p)dV$, can be written in several equivalent forms:

$$2\delta W_1 = \int\{Q^2+\vec{\xi}\cdot\vec{Q}\times\vec{J}+(5/3)p(\mathrm{div}\vec{\xi})^2+(\vec{\xi}\cdot\nabla p)(\mathrm{div}\vec{\xi})\}dV+\oint[\partial p_*/\partial n]\xi_n{}^2 dS, \quad (1)$$

$$2\delta W_2 = \int\{(\vec{B}\cdot\nabla\vec{\xi}-\vec{B}\,\mathrm{div}\vec{\xi})^2+\nabla p_*\cdot(\vec{\xi}\,\mathrm{div}\vec{\xi}-\vec{\xi}\cdot\nabla\vec{\xi})+(5/3)p(\mathrm{div}\vec{\xi})^2\}dV, \quad (2)$$

$$2\delta W_3 = \int\{(\vec{B}\cdot\nabla\vec{\xi}-\vec{B}\,\mathrm{div}\vec{\xi})^2+p_*(\partial\xi_i/\partial x_j)(\partial\xi_j/\partial x_i)+[(5/3)p-p_*](\mathrm{div}\vec{\xi})^2\}dV+\oint p_*[\vec{\xi}\,\mathrm{div}\vec{\xi}-\vec{\xi}\cdot\nabla\vec{\xi}]dS, \quad (3)$$

where

$$\vec{Q} = \mathrm{curl}(\vec{\xi}\times\vec{B}), \quad (4)$$

$$p_* = p+\tfrac{1}{2}B^2. \quad (5)$$

The surface integrals are kept in order to allow discontinuities in $\vec{B}$ and p at selected flux surfaces; equilibrium requires that p_* be continuous across these surfaces,

$$[p_*] = 0. \quad (6)$$

The same notation is used in (1), where $[\partial p_*/\partial n]$ signifies the jump in $\partial p_*/\partial n$ across an interface, and similarly in (3).

The first form, δW_1, explicitly exhibits the stability of a vacuum field for which $\delta W_1 = \frac{1}{2}\int Q^2 dV$ [this elementary fact is quite hidden in (2) and (3)]. The neutral variations in a vacuum field are interchanges, $\vec{Q} = \mathrm{curl}(\vec{\xi}\times\vec{B}) = 0$.

The second form, δW_2, can be obtained from (1) by innumerable integrations by parts, or more directly, via a Lagrangian (rather than Eulerian) evaluation of the energy variation.[1] This form is notable in that there is no explicit contribution at an interface. It also explicitly exhibits the stability of the elementary configuration in which $\vec{B}$ is unidirectional [B_z and p depend on (x,y) and $p_* = \mathrm{const}$]. This elementary result is not apparent from δW_1. The neutral perturbations in this case are flutes, $\mathrm{div}\vec{\xi} = 0$ and $\vec{B}\cdot\nabla\vec{\xi} = 0$. We can contrast flutes which are incompressible in physical space with interchanges which are incompressible in flux coordinates.

The third form, δW_3, can be obtained from (2) by a single integration by parts. We shall find this form particularly useful because no derivatives of the equilibrium quantities $\vec{B}$ and p appear explicitly.

In two-dimensional equilibrium with z as an ignorable coordinate, the field can be written

$$\vec{B} = \vec{B}_0+\vec{B}_z = \vec{n}\times\nabla\psi+\vec{n}B_z, \quad (7)$$

$$\vec{J} = \vec{n}\Delta\psi-\vec{n}\times\nabla B_z, \quad (8)$$

where $\vec{n}$ is the unit vector in the z direction. The most general equilibrium with this symmetry requires p and B_z to be constant on ψ lines, and ψ is governed by the nonlinear elliptic equation

$$\Delta\psi = -p'(\psi)-f'(\psi), \quad (9)$$

where $p(\psi)$ and

$$f(\psi) = \tfrac{1}{2}B_z{}^2 \quad (10)$$

are considered to be arbitrarily given functions.[2]

We shall be concerned with a special class of equilibria such that

$$p(\psi)+f(\psi)=p+\tfrac{1}{2}B_z^{\,2}=\text{const}, \tag{11}$$

and

$$\Delta\psi=0. \tag{12}$$

Such special equilibria can be constructed by choosing an arbitrary two-dimensional vacuum field $\vec{B}_0=\vec{n}\times\nabla\psi$, then supplying an arbitrary (positive) pressure profile $p(\psi)$ and with it a corresponding $B_z(\psi)$ compatible with (11). Alternatively, special solutions are characterized by $J_z=0$; only plane currents flow in the plasma.

Our main result is that all such special equilibria are absolutely stable, $\delta W>0$. Stability is meant with respect to general three-dimensional perturbations $\vec{\xi}$. There is no restriction at all on the plane vacuum field or on the pressure profile $p(\psi)$. Examples of relevant fields are the simple hard core, any multipole with simple or split axis, Fig. 1, the two-dimensional model of a helical field,[3] Fig. 2, and many more.

We present two proofs of this stability theorem. The first is by inspection of δW_1 and δW_3, but it loses track of the neutral displacements, $\delta W=0$. Consider first a step-function pressure profile; i.e., select a sequence of flux contours $\psi=\psi_i$ in the vacuum field and assign a constant value of p to each shell $\psi_i<\psi<\psi_{i+1}$. There is, of course, a surface current in each ψ_i. Consider δW_1. We have $\partial p_*/\partial n=\partial(\frac{1}{2}B_0^{\,2})/\partial n$ which is continuous across an interface. Therefore the surface terms in δW_1 vanish. In each layer we have $\vec{J}=0$ and $p=0$; thus

$$\delta W_1=\tfrac{1}{2}\int\{Q^2+(5/3)p(\operatorname{div}\vec{\xi})^2\}dV\geq 0. \tag{13}$$

This proof of the stability of step-function special equilibria has been given previously.[3]

Next consider a specific field $\psi(x,y)$ and a smooth profile $p(\psi)$ together with a fixed smooth perturbation $\vec{\xi}(x,y,z)$. We can uniformly approximate the function of a single variable $p(\psi)$ by a sequence of step functions $p^n(\psi)$. The corresponding step functions $B_z^{\,n}(\psi)$ also converge uniformly to $B_z(\psi)$. Since ψ is a smooth function of x and y, p^n and $B_z^{\,n}$ converge to their respective limits as functions of x and y. For each p^n we have $\delta W^n\geq 0$. By inspection of formula (3) we conclude that δW^n converges to δW ($\vec{B}_0$ and $\vec{\xi}$ are fixed, and the integral contains no derivatives of p or B_z). Consequently $\delta W\geq 0$ for a smooth $p(\psi)$ for every $\vec{\xi}$. We repeat that, although the equiibrium is two-dimensional, we have proved its stability with respect to general three-dimensional perturbations $\vec{\xi}$.

Our second proof uses the following form of δW which is valid only for the special class of equilibria, (11):

$$2\delta W_4=\int\{(\vec{B}\cdot\nabla\vec{\xi}-\vec{B}\operatorname{div}\vec{\xi}-\xi\cdot\nabla\vec{B}_0)^2+(5/3)p(\operatorname{div}\vec{\xi})^2\}dV. \tag{14}$$

Stability follows by inspection of δW_4. This form for δW is derived from (1) or (2) by an intricate series of manipulations involving chiefly the identity

$$\operatorname{div}[(\vec{B}_0\times\vec{\xi})\times(\vec{\xi}\cdot\nabla\vec{B}_0)]=(\nabla\tfrac{1}{2}B_0^{\,2})(\vec{\xi}\operatorname{div}\vec{\xi}-\vec{\xi}\cdot\nabla\vec{\xi})+(\vec{B}_0\cdot\nabla\vec{\xi}-\vec{B}_0\operatorname{div}\vec{\xi})^2-(\vec{B}_0\cdot\nabla\vec{\xi}-\vec{\xi}\nabla\vec{B}_0-\vec{B}_0\operatorname{div}\vec{\xi})^2. \tag{15}$$

It is suggestive to consider the positive first term of (14) as an interpolation between an interchange

$$\vec{Q}=\vec{B}\cdot\nabla\vec{\xi}-\vec{B}\operatorname{div}\vec{\xi}-\vec{\xi}\cdot\nabla\vec{B}, \tag{16}$$

and the flute term, $\vec{B}\cdot\nabla\vec{\xi}-\vec{B}\operatorname{div}\vec{\xi}$ of (2), just as the equilibrium is, in a sense, an interpolation between a vacuum field $\vec{B}_0$ and a unidirectional equilibrium $(p,\vec{B}_z)$.

A small amount of manipulation (cf. Appendix) shows that the only neutral perturbation, $\delta W_4=0$,

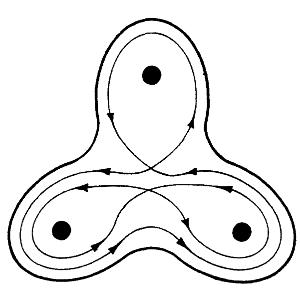

FIG. 1. Multipole fields.

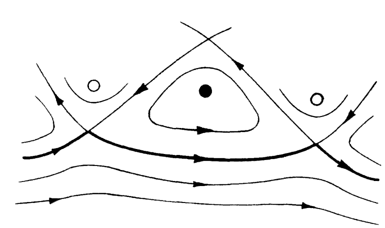

FIG. 2. Model of a helical field.

is the trivial one,

$$\vec{\xi} = \alpha(\psi)\vec{B}_0 + \beta(\psi)\vec{n}, \tag{17}$$

which leaves the equilibrium unchanged. Therefore δW is strictly positive for any nonzero perturbation of p and $\vec{B}$.

Of particular interest is a multipole geometry, cf. Fig. 1. Any such configuration is stable provided that $J_z = 0$. We repeat that the vacuum field, $\vec{B}_0$, is arbitrary; the pressure profile $p(\psi)$ is arbitrary, in particular it is not necessarily monotone; B_z can go through zero and reverse; there is no critical β, and there is no critical flux surface (as in simple multipoles with J_z but no B_z). The fact that δW is strictly positive, with no neutral perturbations, indicates that perturbations of these equilibria to include small J_z, slight helical or toroidal curvature, etc., are likely to yield substantial regions of stability.

The stability of this class of equilibria cannot be even qualitatively related to any curvature, or magnetic well, or mean magnetic well concept. All of these are based on the idea that the plasma would rather be in one location than in another; but we are free to change the sign of $p'(\psi)$ without losing stability. With finite β, $p(\psi)$ can alter the well configuration, but such a self-induced well is excluded in the usual formulations of this principle.

Shear is present in varying amounts in most of these equilibria. But no shear devotee would expect this mechanism to be effective at arbitrary β. Also, the shear can always be reduced to zero in finite volume by proper choice of $p(\psi)$. This geometry is too restricted to permit arbitrary variation of shear, $p(\psi)$, and some selected well criterion all independently. It is unlikely that any weighted combination of shear and some specified well depth extracted from this geometry would be relevant to any but a neighboring geometry (just as these concepts, originating in other special calculations, fail in the present application). It has always been clear that quantitative magnetohydrodynamic stability is a matter of detailed calculation in any given configuration; but even the qualitative picture seems to depend on as yet undescribed features of the equilibrium. Any new qualitative insights given by this large class of stable configurations will only become visible when extensions to more general, neighboring geometries indicate a transition from stability to instability.

Appendix: Neutral displacements.—To find all displacements $\vec{\xi}$ which satisfy

$$\operatorname{div} \vec{\xi} = 0, \tag{A.1}$$

$$\vec{B}\cdot\nabla\vec{\xi} - \vec{\xi}\cdot\nabla\vec{B}_0 = \vec{Q} + \vec{\xi}\cdot\nabla\vec{B}_z = 0, \tag{A.2}$$

we set

$$\vec{\xi} = \alpha\vec{B}_0 + \beta\vec{n} + \gamma\nabla\psi, \quad \vec{\xi}_0 = \alpha\vec{B}_0 + \gamma\nabla\psi. \tag{A.3}$$

We consider a "toroidal" domain, periodic in z, and with either closed or periodic ψ lines.

From the z component of (A.2) we find $\vec{B}\cdot\nabla\beta = 0$ or

$$\beta = \beta(\psi) \tag{A.4}$$

(except when there is a finite volume of closed magnetic lines). From $\operatorname{div}\vec{Q} = 0$ we conclude (unless B_z is identically constant)

$$\gamma = \gamma(x, y). \tag{A.5}$$

Noting that $\partial\vec{Q}/\partial z = 0$, we set $(\partial\vec{\xi}/\partial z)\times\vec{B} = \nabla\varphi$ and obtain

$$B_z(\partial\alpha/\partial z)\nabla\psi = \nabla\varphi. \tag{A.6}$$

Therefore $\varphi = \varphi(\psi)$ and $\alpha = \alpha_0(\psi)z + \alpha_1(x, y)$. From periodicity in z, we conclude

$$\alpha = \alpha(x, y). \tag{A.7}$$

The plane component of (A.2) gives $\operatorname{curl}(\vec{\xi}\times\vec{B}_0) = 0$ or $\gamma B_0^2\vec{n} = \nabla\Phi$, from which

$$\gamma B_0^2 = \sigma = \text{const}, \tag{A.8}$$

$$\vec{\xi}_0 \times \vec{B}_0 = \sigma\vec{n}. \tag{A.9}$$

The condition $\operatorname{div}\vec{\xi} = 0$ or

$$\vec{B}_0\cdot\nabla\alpha + \sigma\nabla\psi\cdot\nabla(1/B_0^2) = 0 \tag{A.10}$$

is compatible with α being single-valued on a ψ contour only if $\sigma = 0$ (assuming that $\oint ds/B_0$ is not identically constant), from which $\vec{B}_0\cdot\nabla\alpha = 0$, and finally

$$\gamma = 0, \tag{A.11}$$

$$\alpha = \alpha(\psi). \tag{A.12}$$

This confirms Eq. (17) in the text.

The special cases excluded above (B_z constant, all B lines closed, $\int ds/B_0$ constant) give certain unimportant exceptions.

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*Work supported by the U. S. Atomic Energy Commission under Contract No. AT(30-1)1480.

†On leave from the Institut für Plasmaphysik, Garching bei München, Germany.

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SELF-FOCUSING OF A PLASMA WAVE ALONG A MAGNETIC FIELD

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It is shown that at frequencies approximately equal to one-half the electron cyclotron frequency, the threshold power for the self-focusing of the whistler mode becomes sufficiently small to be available for laboratory experiments.

In this paper we shall show that at frequencies about one-half the electron cyclotron frequency, it is possible for whistler wave to be self-focused by exceedingly small threshold power.[1] Since we shall be concerned with an order estimation, it may be assumed that electrons and ions are isothermal and governed by hydrodynamic equations. In addition, the frequency of the wave will be taken as much higher than the ion-cyclotron frequency so that the ion motion will be neglected. Then we may take as our starting equations the hydrodynamic equations for electrons coupled with the Maxwell equations. The equations for ions will be supplemented, upon occasion, to determine a slowly varying mode of density.

We first consider a wave of slab shape propagating in the x direction oriented along an applied magnetic field and stretching in the y direction. In this case the starting equations take the form

$$\frac{\partial \mathcal{V}}{\partial t}-i\omega_c\mathcal{V}+\frac{e}{m}\mathcal{E}=-\frac{\kappa T_e}{m(n_0+n)}\frac{\partial n}{\partial y}-u\frac{\partial \mathcal{V}}{\partial x}-v\frac{\partial \mathcal{V}}{\partial y}+i\frac{\omega_c}{B_0}(B_x\mathcal{V}-u\mathcal{B}), \tag{1}$$

$$\frac{\partial \mathcal{B}}{\partial t}+ic\frac{\partial \mathcal{E}}{\partial x}=ic\frac{\partial E_x}{\partial y}. \tag{2}$$

$$\frac{\partial \mathcal{E}}{\partial t}-ic\frac{\partial \mathcal{B}}{\partial x}-4\pi e n_0\mathcal{V}=-ic\frac{\partial B_x}{\partial y}+4\pi e n\mathcal{V}, \tag{3}$$

$$\frac{\partial n}{\partial t}+n_0\frac{\partial u}{\partial x}=-n_0\frac{\partial v}{\partial y}-\frac{\partial(nu)}{\partial x}-\frac{\partial(nv)}{\partial y}. \tag{4}$$

$$\frac{\partial u}{\partial t}+\frac{e}{m}E_x=-\frac{\kappa T_e}{m(n_0+n)}\frac{\partial n}{\partial x}-u\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial y}-\frac{\omega_c}{B_0}\mathrm{Im}(\mathcal{V}^*\mathcal{B}), \tag{5}$$

$$\frac{\partial B_x}{\partial t}+c\frac{\partial E_z}{\partial y}=0, \tag{6}$$

$$\frac{\partial E_x}{\partial t}-4\pi e n_0 u=c\frac{\partial B_z}{\partial y}+4\pi e n u. \tag{7}$$

$$\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}=0, \tag{8}$$

$$\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}=4\pi e(n_i-n), \tag{9}$$

$$\mathcal{V}=v+iw,\quad \mathcal{E}=E_y+iE_z,\quad \mathcal{B}=B_y+iB_z.$$

Here u, v, and w are the x, y, and z components of the velocity of electrons, respectively; n is the varying part of the total density of electrons n_0+n. E_x, E_y, E_z, B_y, and B_z denote the respective components of the electric and magnetic field while the x component of the magnetic field is the sum of the applied strength B_0 and the varying part B_x, and ω_c is the electron cyclotron