phase, are planned.

†The research reported in this paper has been made possible through the support of the Science Research Council who also provided a maintenance grant for one of us (J.M.H.).

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PERIODIC PULLING AND TURBULENCE IN A BOUNDED PLASMA*

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Experimental results obtained with a Q machine demonstrate the effect of a nonlinear mechanism called periodic pulling that has been proposed to explain the transition to turbulence in a bounded plasma characterized by weakly unstable modes. The effect is characteristic of any distributed-parameter system that can be described by the van der Pol equation and is related to phenomena observed in solid-state oscillators and the gas laser.

The transition from laminar flow to turbulence in a physical system is generally taken to imply the introduction of new degrees of freedom in the form of new modes of oscillation at incommensurate frequencies.[1] However, certain questions arise when this concept is applied to experimental observations of drift instabilities in a Q machine. At low instability levels the frequency spectrum consists of discrete, harmonically related frequencies that correspond to various azimuthal mode numbers. At slightly higher amplitudes the spectrum suddenly becomes more or less continuous, as is characteristic of turbulence.[2] There is some question, however, as to how this transition can occur. The unstable plasma is a bounded system with discrete modes, and at the weakly nonlinear levels typical of the experiments it would be expected that the spectrum could be resolved into commensurate components characteristic of the initial mode frequencies, which are related as a series of ascending integers.

It has been proposed that the introduction of incommensurate frequencies into the spectrum can be explained by a nonlinear mechanism called periodic pulling,[3] which is essentially the incomplete entrainment of a mode by a perturbation whose frequency and amplitude are just beyond the range at which the mode would be locked to the perturbation. Under these conditions the mode is frequency and amplitude modulated, producing a multiplicity of sidebands at frequencies incommensurate with the initial mode frequency. Periodic pulling is characteristic of any system that can be described by the van der Pol equation.

This Letter reports on recent experimental results that support the explanation given above. A preliminary account has been given earlier.[4]

The analysis makes use of a model in which the modes of oscillation in a weakly nonlinear unstable plasma are represented as an ensemble of weakly coupled van der Pol oscillators, with the spatial parts of the mode functions being identical to those of the linear case.[3] Therefore, we consider a perturbed mode as represented by the van der Pol equation with a harmonic forcing term:

$$\frac{d^2x}{dt^2}-\epsilon(1-\beta x^2)\omega_0\frac{dx}{dt}+\omega_0^2x=\omega_0^2E\cos\omega_it, \qquad (1)$$

where the perturbation amplitude E is small in a sense described below, and the perturbation frequency ω_i is approximately equal to the free-running frequency ω_0. Here, ϵ is a small parameter that characterizes the nonlinearity and $\beta\equiv 3\delta/\gamma$, where γ is the linear growth rate and δ is associated with the saturation mechanism. The quantity x can represent any characteristic perturbation variable, say the ac density or potential. If the solution of Eq. (1) is written

$$x(t)=A(t)\sin[\omega_it-\varphi(t)], \qquad (2)$$

where $A(t)$ and $\varphi(t)$ are slowly varying functions of time, then the amplitude and phase obey the following coupled equations[3,5]:

$$dA/dt = \epsilon f(x,\dot{x})A + \mu(E/2)\omega_0 \cos\varphi, \qquad (3)$$

$$d\varphi/dt = \mu[\Omega_0 - (E/2A)\omega_0 \sin\varphi], \qquad (4)$$

where $\Omega_0 \equiv \omega_i - \omega_0$, $f(x,\dot{x}) \equiv \omega_0(1-\beta x^2)\dot{x}$, and ϵ and μ are smallness parameters that satisfy the relation $\mu \ll \epsilon \ll 1$, but $\mu > \epsilon^2$. When $\mu = 0$ the usual van der Pol solution is obtained, $A = A_0$.

In the present analysis we assume $E/A = O(\mu)$ and $\Omega_0/\omega_0 = O(\mu)$. Physically this means that we are treating small adiabatic variations on a time scale μt about the steady-state oscillations of amplitude A_0 and frequency ω_0. We note that the condition $\varphi = 0$ can be satisfied when $\alpha \equiv E\omega_0/2A\Omega_0 > 1$; this corresponds to the mode-locking condition that has been described earlier.[2] In the present work we are interested specifically in values $\alpha < 1$.

Equations (3) and (4) imply that the perturbed van der Pol mode is subject to simultaneous amplitude and frequency modulation on a time scale μt. The quantity φ in (3) and (4) is given by[3]

$$\varphi = \Omega_0' t + 2\sum_{n=1}^{\infty} \frac{\kappa^n}{n} \sin n\Omega_0' t, \qquad (5)$$

where $\Omega_0' \equiv (1-\alpha^2)^{1/2}\Omega_0 t$ and $\kappa \equiv [(1-\alpha^2)^{1/2}-1]/[(1-\alpha^2)^{1/2}+1]$.

The simultaneous amplitude and frequency modulation by the complex function in (5) represents a case of multitone modulation, which can produce an extremely complicated frequency spectrum. However, under typical experimental conditions $\kappa \ll 1$ and only the Fourier components of φ corresponding to the first few values of n need be considered. The combined AM-FM sideband spectrum for the $n = 1$ component of φ is given by[3]

$$x_1(t) = \sum_{m=-\infty}^{\infty} J_m(\kappa)\left[1 - m\frac{M}{\kappa}\right] \times \exp[i(\omega_0 + m\Omega_0')t], \qquad (6)$$

where M is the effective amplitude-modulation index ($M \ll 1$). Since $\kappa \ll 1$ we need only consider the sideband component corresponding to $m = \pm 1$ and (6) shows that sideband cancellation can occur on one side of the unperturbed frequency under appropriate conditions, i.e., $M \sim \kappa$ and $m = +1$. Moreover the nonvanishing sideband will always lie on the side of ω_0 away from the perturbation frequency ω_i. This is the case since (6) shows that when $M \sim \kappa$ sideband cancellation occurs for $\omega = \omega_0 + |\Omega_0'|$ when $\Omega_0' > 0$ and $\omega = \omega_0 - |\Omega_0'|$ when $\Omega_0' < 0$. Equation (6) describes the effect of a single Fourier component of the modulation function in (5). However, the results hold for all of the Fourier components (see Fig. 1) and a key feature of the analysis is the prediction that reversal of the relative positions of ω_0 and ω_i should cause the sideband structure to reflect about ω_0. Equation (5) also implies that the logarithm of the sideband amplitude is a linear function of frequency.[3]

The sideband cancellation effect is actually an application of the so-called analytic-signal theorem,[6] in which a real function of time $s(t)$ is used to form a complex function $S(t) = s(t) + i\sigma(t)$. If its real and imaginary parts are reciprocal Hilbert transforms, the complex function $S(t)$ is analytic and the Fourier transform of $S(t)$, which we denote by $\mathfrak{F}\{S(t)\}$, satisfies the condition[6]

$$\mathfrak{F}\{S(t)\} \neq 0, \quad \omega > 0;$$

$$\mathfrak{F}\{S(t)\} = 0, \quad \omega < 0.$$

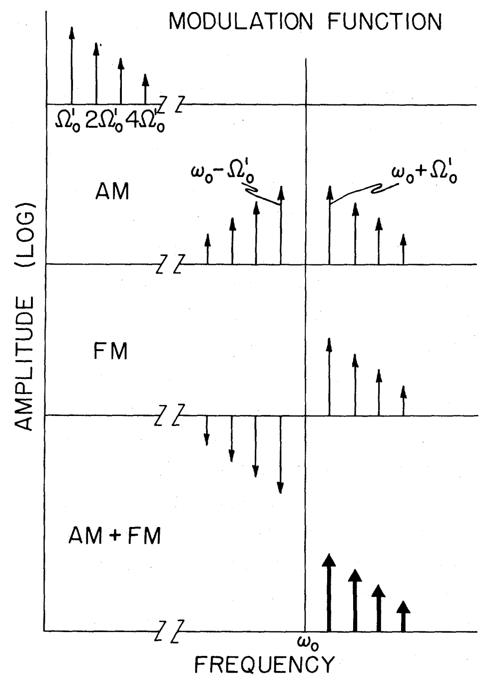

FIG. 1. Diagram to show sideband cancellation in combined amplitude and frequency modulation. The top spectrum corresponds to the effective modulation function, with components Ω_0', $2\Omega_0'$, etc. In the approximation used in the text each such component produces one pair of amplitude-modulation sidebands $\omega_0 \pm \Omega_0'$, $\omega_0 \pm 2\Omega_0'$, etc., and one pair of frequency-modulation sidebands $\omega_0 \pm \Omega_0'$, $\omega_0 \pm 2\Omega_0'$, etc., with the relative phase relations as shown in the two center spectra. The resultant asymmetric spectrum is shown at the bottom. This pattern applies for $\Omega_0' < 0$.

The converse relation can also be demonstrated. That is to say, a single-sided frequency spectrum (with respect to the original unperturbed frequency) can only be produced by a time function of the form $S(t) = s(t) + i\sigma(t)$, where $s(t)$ and $\sigma(t)$ are reciprocal Hilbert transforms, as is the case for $\sin\varphi$ and $\cos\varphi$ in Eqs. (3) and (4).

It is interesting to note that the analytic-signal theorem is the counterpart of the well-known Kramers-Kronig relations,[7] with the roles of the time and frequency variables being interchanged. The quantity $S(t)$ plays the role of the response function in the Kramers-Kronig relations and the notion of causality or one-sidedness in time is replaced by the one-sided frequency spectrum.[3]

The results of experiments carried out in connection with the present analysis are shown in Fig. 2. These experiments were performed in a single-ended Q machine[8] in which a neutral potassium flux is directed at a tantalum disk heated to 2000°K to produce a low-density ($n < 10^9$ cm^{-3}) plasma in a magnetic field. The oscillations in the plasma potential due to the drift instability are detected with a floating probe and displayed on a spectrum analyzer. In these experiments a weak signal from a conventional audio oscillator is coupled to the cold end plate to simulate the effect of a weak perturbation in the plasma and the parameter α, discussed above, is somewhat smaller than unity. A typical unperturbed mode is shown in the top trace in Fig. 2 while the perturbed modes are shown in the center and bottom traces for the cases $\Omega_0' > 0$ and $\Omega_0' < 0$, respectively. The spectrum reversal effect described above is clearly evident, as is the linear dependence of the logarithm of the amplitude on frequency.

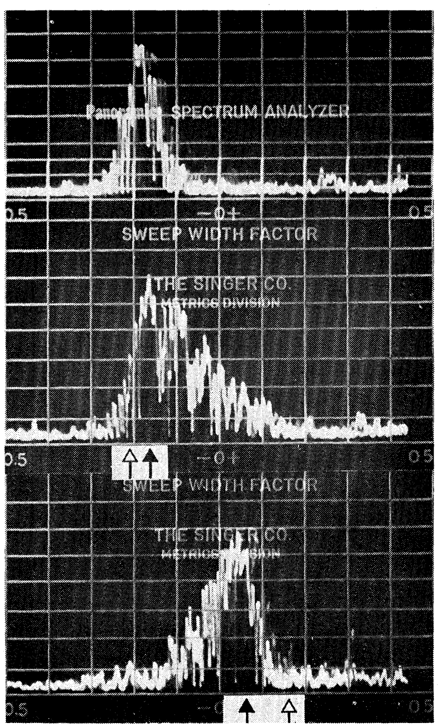

FIG. 2. Experimental results to show the reversal of the sideband spectrum as a function of the sign of $\Omega_0' \equiv \text{const} \times (\omega_i - \omega_0)$ where ω_0 is the unperturbed mode frequency and ω_i is the frequency of the perturbation. Logarithmic vertical scale; total frequency width = 2 kHz. Upper trace: unperturbed mode. Center trace: unperturbed mode at 3.115 kHz (solid arrow), perturbation at 3.205 kHz (hollow arrow), $\Omega_0' > 0$. Bottom trace: unperturbed mode at 3.136 kHz (solid arrow), perturbation at 2.900 kHz (hollow arrow), $\Omega_0' < 0$. In both cases the perturbation amplitude is approximately 20 dB below that of the mode. In these pictures the frequency increases from right to left.

The conditions in these experiments are highly specialized and have been set up to verify the operation of the periodic-pulling mechanism. In an actual unstable plasma the role of the perturbing signal would be played by weak oscillations due to higher order nonlinear interactions between unstable drift modes. These weak oscillations can exhibit second-order frequency shifts due to Larmor-radius and finite-length corrections, thus providing the frequency displacement that leads to periodic pulling rather than mode locking.[3]

Periodic pulling provides a means of introducing incommensurate frequencies into a spectrum of discrete modes of the drift instability and these incommensurate frequencies can be translated to the region near zero frequency by nonlinear mixing effects. The net result is the appearance of low-frequency transverse electric fields that can play a role in plasma diffusion.

The results described here provide further support for the van der Pol model of unstable drift modes in a bounded plasma and appear to be relevant to other systems described by the van der Pol equation. For example, similar phenomena have been observed in beam-plasma interactions,[9] in avalanche-diode oscillators,[10] and in a He-Ne laser in a weak magnetic field.[11]

We are indebted to W. E. Crowe and R. M. Monblatt for help in these experiments.

*Work supported by the U. S. Air Force Office of Scientific Research [AF-1016-68] and the U. S. Atomic Energy Commission [AT-(40-1)-3405].

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THEORY OF DILUTE SOLUTIONS OF He^3 IN LIQUID He^4 †

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We present a microscopic calculation of the effective mass and interactions of He^3 quasiparticles in dilute He^3-He^4 solutions. The theory is first-principles in the sense that it requires no other information than the bare masses of the helium atoms and a realistic two-particle potential. In this note we report our findings in the limiting case of zero concentration.

The properties of a quantum fluid at low temperatures are determined by the elementary excitations of the system. For a dilute solution of He^3 in liquid He^4, the low-lying excitations are of two types: phonons originated in liquid He^4 modified by the isotopic impurities, and a He^3 single-particle spectrum modified by the boson medium. The phenomenological theory of Landau and Pomeranchuk[1] predicted that the He^3 constituents in the solution behave as a Fermi liquid. Recent papers[2] by Bardeen, Baym, and Pines and co-workers and by Emery showed that one can indeed estimate from experimental data the Fermi-liquid parameters: the effective mass of the He^3 quasiparticles and the effective interactions between them.

We have succeeded in constructing a microscopic theory capable of calculating these parameters from bare atoms and realistic two-particle potentials. The program consists of two parts. One part concerns a complete solution of the ground state of the binary boson mixture. The other part assumes this solution and constructs a correlated basis making full use of available physical information, and develops a perturbation theory in the correlated representation. This note deals with the second part. The information required from the first part can be briefly summarized as follows.

For a system comprising N_3 mass-3 bosons and $N_4 \equiv N - N_3$ mass-4 bosons, interacting pairwise via a potential $v(r)$, the Hamiltonian is given by

$$H = \sum_{i=1}^{N_3} \frac{-\hbar^2}{2m_3}\nabla_i^2 + \sum_{j=N_3+1}^{N} \frac{-\hbar^2}{2m_4}\nabla_i^2 + \tfrac{1}{2}\sum_{\substack{k \neq l \\ =1}}^{N} v(r_{kl}), \tag{1}$$

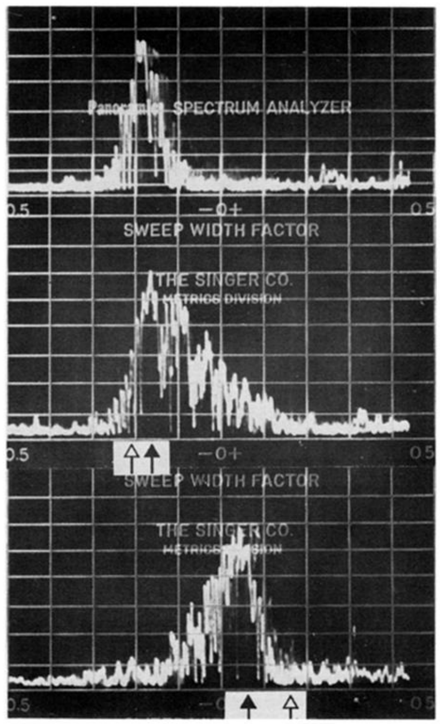

FIG. 2. Experimental results to show the reversal of the sideband spectrum as a function of the sign of Ω_0' $\equiv \text{const} \times (\omega_i - \omega_0)$ where ω_0 is the unperturbed mode frequency and ω_i is the frequency of the perturbation. Logarithmic vertical scale; total frequency width = 2 kHz. Upper trace: unperturbed mode. Center trace: unperturbed mode at 3.115 kHz (solid arrow), perturbation at 3.205 kHz (hollow arrow), $\Omega_0' > 0$. Bottom trace: unperturbed mode at 3.136 kHz (solid arrow), perturbation at 2.900 kHz (hollow arrow), $\Omega_0' < 0$. In both cases the perturbation amplitude is approximately 20 dB below that of the mode. In these pictures the frequency increases from right to left.