GENERAL PARAMETRIZATION OF TRAJECTORY AND RESIDUE FUNCTIONS FOR DAUGHTER REGGE POLES*

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We present the most general parametrizations for the trajectory and residue functions for daughter Regge poles which are consistent with analyticity and unitarity. Our results are confined to the scattering of spinless particles, but are valid for arbitrary external masses and nonparallel trajectories. We show that except in the special case of pairwise-equal external masses, an infinite number of Lorentz poles contribute to the scattering amplitude at $s=0$. We obtain a simple parametrization of the scattering amplitude for $s\sim 0, t\to\infty$, which involves powers of both t and $\ln t$.

Freedman and Wang[1] showed some time ago that any Regge pole which contributes to an s-channel scattering process $1+2\to 3+4$ in which $m_1\neq m_2$ and $m_3\neq m_4$ must be accompanied by an infinite set of daughter Regge poles. The behavior of the trajectory and residue functions for the daughter poles is restricted near $s=0$ by the condition that apparent singularities in the individual terms in the Regge expansion cancel in the full scattering amplitude. In particular, successive daughter poles have opposite signatures, trajectory functions $\alpha_n(s)$ which are spaced by integers at $s=0$, $\alpha_n(0)=\alpha_0(0)-n$, $n=1,2,\cdots$, and reduced residue functions $\bar{\beta}_n(s)$ which diverge as s^{-n} for $s\to 0$. Although several authors[2] have derived the first few terms in the Taylor series expansions of $\alpha_n(s)$ and $s^n\bar{\beta}_n(s)$ about $s=0$, rather little has been known about the general form of these functions.[3,4]

In this paper, we give the most general parametrizations of the daughter trajectory and residue functions which are consistent with the requirements of analyticity and unitarity (factorization of the Regge residues). The present results are confined to the scattering of spinless particles, but are valid for arbitrary external masses and nonparallel daughter trajectories. Except in the special case of pairwise-equal external masses ($m_1=m_2$ and $m_3=m_4$), we find that the set of Regge poles in a parent-daughter sequence necessarily corresponds at $s=0$ to a reducible representation of the Lorentz group, rather than to an irreducible representation, as is often assumed. This result is of considerable significance for the group-theoretical approach to the classification of Regge poles. Our results have immediate practical applications to the parametrization of daughter Regge trajectories and residues away from $s=0$ and to high-energy phenomenology. We obtain, in particular, a simple parametrization of the scattering amplitude for $s\sim 0$, $t\to\infty$, which involves powers of both t and $\ln t$.

Our construction of daughter trajectory and residue functions is based on the Lorentz expansion for the spinless scattering amplitude.[5] We begin with the expression given in Eq. (10) of Ref. 5,

$$A(s,t,u)=\tfrac{1}{2}\pi\sum\nolimits_n(2\alpha_n^{\pm}+1)\bar{\beta}_n^{\pm}(s)(\sin\pi\alpha_n^{\pm})^{-1}[\Gamma(\alpha_n^{\pm}+1)]^{-2}\sum\nolimits_l(-1)^l$$
$$\times\{[\Gamma(l+1)\Gamma(2\alpha_n^{\pm}-l+1)C_l^{(-\alpha_n^{\pm})}(\cosh\beta_1)C_l^{(-\alpha_n^{\pm})}(\cosh\beta_3)-(l\to l-2)]$$
$$\times D_{-\alpha_n^{\pm}+l-2}^{(1)}(\cosh\beta_t)\pm(m_4/m_1)^{\alpha_n^{\pm}}[1\to 1, 3\to 4, t\to u]\}+\text{background integrals.} \qquad (1)$$

The sums run over all Regge poles to the right of some (arbitrary) line $\mathrm{Re}\,\alpha=-\Gamma-1$ in the complex j plane, and over those values of l for which $\mathrm{Re}\,\alpha_n>-\Gamma+l-1$. The background integrals in this expression are analytic functions of s for $s\to 0$, and do not enter in our argument. The variables $\cosh\beta_t$ and $\cosh\beta_u$ are simply related to the Mandelstam variables t and u, $\cosh\beta_t=[m_1^2+m_3^2-t]/2m_1m_3$, $\cosh\beta_u=[m_1^2+m_4^2-u]/2m_1m_4$. The functions $D_{-\alpha_n+l-2}^{(1)}$ (Gegenbauer functions of the second kind[5]) are analytic functions of s for $s\to 0$ provided the α_n are analytic at that point. As a result, the $s=0$ singularities in the individual terms in (1) are isolated in the Gegenbauer polynomials $C_l^{(-\alpha)}(\cosh\beta_i)$, $\cosh\beta_i = \epsilon_i/m_i$. For $s \to 0$ and unequal external masses, the particle energies diverge $\{$e.g., $\cosh\beta_1 = [s + m_1^2 - m_2^2]/2m_1\sqrt{s}\}$, and the lth term in (1) for a given α_n contains terms which diverge as s^{-l}, $s^{-l+1}, \cdots$. These singularities will appear in $A(s,t,u)$ unless the Regge poles appear in sets with the parent-daughter properties. The problem of constructing the most general daughter trajectory and residue functions is just that of finding those forms for $\alpha_n(s)$ and $\beta_n(s)$ for which all the singularities in (1) cancel at $s=0$ for arbitrary, but finite, values of Γ, and arbitrary, nonparallel trajectories $\alpha_n(s)$. The construction is subject to the condition that the Regge residues $\beta_n(s)$ factor; this condition is a direct consequence of unitarity as applied to the partial-wave amplitudes.[6] The proof that the results for α_n and β_n given in Eqs. (2) and (4) lead to an analytic scattering amplitude involves the substitution of (4) into (1), expansion of the trajectory-dependent functions in powers of $(\alpha_n + n - \alpha_0)$, and the use of two new sum rules for the Gegenbauer functions.[5] Details of the construction of α_n and β_n and of the proof of analyticity will be published elsewhere.

The most general form for the daughter trajectory functions is given by

$$\bar{\alpha}_n(s) = \sum_{j=0}^{n} \Gamma(n+1)\Gamma(2\bar{\alpha}_n(s)+2-n)[\Gamma(n-j+1)\Gamma(2\bar{\alpha}_n(s)+2-n-j)]^{-1} s^j a_j(s). \tag{2}$$

Here $\bar{\alpha}_n(s) = \alpha_n(s) + n$. Note that $\bar{\alpha}_n(0) = \alpha_0(0)$ for all n. The set of functions $a_j(s)$ is fixed; the same functions appear in the expressions for all $\bar{\alpha}_n$.[7] These functions may consequently be determined successively from the set of Eq. (2). A given function a_n will clearly depend on all the trajectory functions a_j for $j<n$. The a_j must be analytic at $s=0$ if $A(s,t,u)$ is to be analytic at that point. This condition leads to the relations among the derivatives of the α_n at $s=0$ characteristic of daughter Regge trajectories. In particular, the mth derivative of α_n at $s=0$ can be expressed in terms of the first m derivatives of the α_j with $j<m$, provided that $m<n$. The nth and higher derivatives of α_n are arbitrary. The results previously available[2] include only expressions for $\bar{\alpha}_n'(0)$ and $\bar{\alpha}_n''(0)$; these agree with the expressions derived from (2).

Equation (2) contains all the information on the behavior of the daughter-trajectory functions which can be derived from the requirement that $A(s,t,u)$ be analytic at $s=0$. We obtain a stronger result by using two additional facts: (i) The trajectory functions α_n are real analytic functions of s with only dynamical right-hand cuts. (ii) The behavior of the imaginary part of α_n at the physical s-channel threshold is different for different threshold values of $\mathrm{Re}\,\alpha_n(s)$.[6] Since the α_n are all different at $s=0$, these observations lead immediately to the important results that the daughter trajectories cannot be strictly parallel to the leading trajectory (or to each other) and that the a_j cannot be identically zero. It is then easily seen from (2) that the mth derivative of $\alpha_n(s)$ at $s=0$ increases in general for large n $(n \gg m)$ as n^{2m}. As a result, the range of s over which the first few terms in the Taylor series expansion for $\alpha_n(s)$ provide a reasonable approximation for this function shrinks rapidly with increasing n. Phenomenological models based on such expansions must consequently be treated with great care.[8]

The functions $\beta_n(s)$ are somewhat more complicated. We obtain

$$\beta_n(s) = (4\sinh\beta_1 \sinh\beta_3)^{\alpha_n}\bar{\beta}_n(s) = (4p_1p_3/m_1m_3)^{\alpha_n}\bar{\beta}_n(s), \tag{3}$$

where the reduced residue $\bar{\beta}_n$ is given by

$$\bar{\beta}_n(s) = (-1)^n \left[\frac{\Gamma(n+1)\Gamma(2\bar{\alpha}_n+2)}{\Gamma(2\bar{\alpha}_n+2-n)}\right]\left[\frac{\Gamma(\bar{\alpha}_n-n+1)}{\Gamma(\bar{\alpha}_n+1)}\right]^2 C_n^{(\bar{\alpha}_n-n+1)}(\cosh\beta_1)$$
$$\times C_n^{(\bar{\alpha}_n-n+1)}(\cosh\beta_3) B_n \left\{\sum_{j=0}^{n}\left[\frac{\Gamma(n+1)\Gamma(2\bar{\alpha}_n+2-n)}{\Gamma(n-j+1)\Gamma(2\bar{\alpha}_n+2-n-j)}\right] s^j b_j^{(1,2)}(s)\right\}\{(1,2)\to(3,4)\}. \tag{4}$$

The Gegenbauer functions and the last two factors in (4) describe the coupling of the nth trajectory to the particles at the initial and final vertices. The functions $b_j^{(1,2)}$ and $b_j^{(3,4)}$ must be analytic at $s=0$ and must be adjusted to ensure that $\bar{\beta}_n$ vanishes for α_n a negative integer, but are otherwise arbitrary and independent of the α_n. Their behavior can be determined only by dynamical considerations. The essential features of the factor B_n are determined entirely by the trajectory functions:

$$B_n = \frac{1}{1-g_n'(\bar{\alpha}_n)} \sum_{j=0}^{n} s^j C_j(s) \left[\frac{\Gamma(n+1)\Gamma(2\bar{\alpha}_n+2-n)}{\Gamma(n-j+1)\Gamma(2\bar{\alpha}_n+2-n-j)} \right], \tag{5}$$

$$g_n'(z) = \frac{d}{dz} g_n(z) = \frac{d}{dz} \sum_{j=1}^{n} s^j a_j(s) \left[\frac{\Gamma(n+1)\Gamma(2z+2-n)}{\Gamma(n-j+1)\Gamma(2z+2-n-j)} \right]. \tag{6}$$

The functions a_j in (6) are the same as those which appear in (2) and can be expressed in terms of the $\bar{\alpha}_n$. The functions c_j must be analytic at $s=0$ and must be adjusted to eliminate any poles in B_n associated with zeros of the function $(1-g_n')$. The c_j are otherwise unrestricted. Appropriate c_j are easily constructed. It is easily checked that the B_n are necessarily functions of s and differ for different values of n. This factor was omitted in the forms for $\bar{\beta}_n$ proposed by several authors.[4] Their results for $A(s,t,u)$ are consequently analytic at $s=0$ only in the unphysical case of strictly parallel trajectories, $\bar{\alpha}_n(s) \equiv \alpha_0(s)$, or in the special case of pairwise-equal external masses.

The parametrization for $\bar{\beta}_n$ given in (4) contains all the information on these functions which follows from the analyticity of $A(s,t,u)$ at $s=0$ and the condition that the Regge residues factor. For appropriate choices of the functions $b_j^{(1,2)}$, $b_j^{(3,4)}$, and c_j, $\bar{\beta}_n$ will be a real analytic function of s with only the dynamical right-hand cut and poles of order $n, n-1, \cdots$ at $s=0$.

The result for $\bar{\beta}_n$ has several important features: (i) The factor B_n is universal and appears in $\bar{\beta}_n$ for all reactions which involve the set of trajectories $\{\alpha_n\}$. This result holds also for the couplings of these trajectories (of Lorentz type $j_0=0$) to particles with spin. (ii) The first $n-1$ derivatives of the function $s^n\bar{\beta}_n$ are determined at $s=0$ by the first $n-1$ derivatives of the sets of functions $s^j\bar{\beta}_j$ and α_j, $j<n$. The nth and higher derivatives of $s^n\bar{\beta}_n$ are arbitrary. Since the conditions on the first $n-1$ derivatives must be satisfied if $A(s,t,u)$ is to be analytic at $s=0$, the parametrization (5) is as general as possible. (iii) The lack of symmetry in $\bar{\beta}_n$ associated with the distinguished role of particles 1 and 3 in (4) is only apparent. By appropriate rearrangement of the Gegenbauer functions and the last two factors, the result can be brought into an equivalent form with $\cosh\beta_1$ and $\cosh\beta_3$ replaced, for example, by $\cosh\beta_2$ and $\cosh\beta_4$. (iv) The reduced residues $\bar{\beta}_n$ are analytic functions of the external masses. All (nonzero) mass configurations are encompassed in a single expression. There is no distinction between equal-mass scattering ($m_1=m_2$ and $m_3=m_4$), equal-unequal mass scattering ($m_1=m_2$, $m_2\neq m_4$ or $m_1\neq m_2$, $m_3=m_4$), and the general case ($m_1\neq m_2$, $m_3\neq m_4$) as far as the form of $\bar{\beta}_n$ is concerned. This is not true of the partial results for $\bar{\beta}_n$ obtained by some authors.[2] In these expressions, the general-mass results become singular or change form as the mass differences are reduced to zero.

We can immediately establish several results which are of considerable significance for the Lorentz group approach to the classification of Regge poles at $s=0$. We note first that the terms in (1) can be grouped at $s=0$ according to the orders of the Gegenbauer functions $D_{-\alpha_n+l-2}^{(1)}$. If all terms in (1) vanish at $s=0$ except for the single term proportional to $D_{-\alpha_0-2}^{(1)}$, the entire sequence of parent and daughter Regge poles will correspond to a single Lorentz pole and can be classified as a single irreducible representation of the Lorentz group. If more terms remain, the representation of the Lorentz group will be reducible.[9] We distinguish three cases:

(i) In the general mass case, there are finite pieces in every term which fail to cancel at $s=0$. The finite pieces which arise from B_n are necessarily present. An infinite number of Lorentz poles located at the points $\alpha_0(0)-n, n=0,1,2\cdots$ contribute to the scattering amplitude at $s=0$, and the parent-daughter sequence of Regge poles corresponds to a reducible representation of the Lorentz group.[10] Near $s=0$, the contribution of the parent-daughter sequence to the scattering amplitude assumes the simple form

$$A(s,t,u)=[e^{-i\pi\alpha_0(s)}\pm 1]\exp(\alpha_0(s)\beta_t)[\sinh\beta_t \sin\pi\alpha_0(s)]^{-1}$$
$$\times\sum_n \sum_{j=0}^{n} |\beta_t|^j \exp(-n\beta_t)\gamma_{n,j}(s)+O(s\beta_t^{n+1}). \quad (7)$$

The functions $\gamma_{n,j}(s)$ are independent, essentially arbitrary, and analytic at $s=0$. Only $\gamma_{0,0}$ factors. Since $|\beta_t|\to\ln(t/m_1m_3)$ for $t\gg m_1m_3$, this expansion effectively involves powers of $\ln t$ as well as the usual powers of t. This expansion may be useful for high-energy phenomenology.

(ii) In the case of equal-unequal mass scattering, an infinite sequence of Lorentz poles located at the points $\alpha_0(0)$, $\alpha_0(0)-2$, $\alpha_0(0)-4$, $\cdots$, contributes to the scattering amplitude at $s=0$, and the representation of the Lorentz group is again reducible. In this case, the sum on n in (7) is restricted to even integers, and the sum on j runs only from 0 to $\frac{1}{2}n$.

(iii) For pairwise-equal external masses, only the term in (1) proportional to $D_{-\alpha_0-2}{}^{(1)}$ survives at $s=0$. The Regge-pole sequence therefore corresponds to a single Lorentz pole and classifies as an irreducible representation of the Lorentz group at that point, a result obtained independently by Bronzan and Jones.[11] The scattering amplitude may be approximated for small s by the first term in (7), $n=j=0$.

Since it is only in the equal-mass case that the Lorentz group is a symmetry group of the scattering amplitude at $s=0$, the foregoing results are not unexpected (however, see Ref. 10). We should nevertheless emphasize an unusual feature in case (ii). Equal-mass particles can couple only to a single Lorentz pole at $s=0$ if $m_1=m_2$ and $m_3=m_4$, but couple to an infinite number of Lorentz poles in equal-unequal mass scattering. This rather startling result is a consequence of the fact that the residues of the Lorentz poles need not, and in general do not, factor.[5]

After this paper was completed, we received preprints from Professor J. C. Taylor, Oxford University, and J. B. Bronzan, Massachusetts Institute of Technology, which contain some of the above results.

We would like to thank the Aspen Center for Physics for the hospitality accorded us during the summer of 1968, when part of this work was performed. Conversations at that time with Professor Stanley Klein on his approach to the residue problem were particularly appreciated.

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[1]D. Z. Freedman and J. M. Wang, Phys. Rev. Letters 17, 569 (1966), and Phys. Rev. 153, 1596 (1967).

[2]L. Durand, III, Phys. Rev. 154, 1537 (1967); G. Domokos and P. Suranyi, Nuovo Cimento 56A, 445 (1968), and 57A, 813 (1968); L. Jones and W. K. Shepard, California Institute of Technology Report No. CALT-68-157, 1968 (unpublished); P. DiVecchia and F. Drago, Phys. Letters 27B, 387 (1968), and California Institute of Technology Reports No. CALT-68-172 and No. CALT-68-176, 1968 (unpublished); J. B. Bronzan, C. E. Jones, and P. K. Kuo, to be published; J. B. Bronzan, to be published; S. A. Klein, to be published; P. K. Kuo, to be published. Coefficients in the expansion for α_n given by Domokos and Suranyi are not all independent.

[3]Although general expressions have been suggested for the daughter residue functions (cf. Ref. 4), these are either restricted to the case of equal-mass scattering or are based on unphysical assumptions, e.g., that the parent and daughter trajectories are strictly parallel, or that the entire set of trajectories can be classified at $s=0$ as a single irreducible representation of the Lorentz group.

[4]D. Z. Freedman and J. M. Wang, Phys. Rev. 153, 1596 (1967), and 160, 1560 (1967); G. Domokos and P. Suranyi, Nucl. Phys. 54, 529 (1964); K. M. Bitar and G. L. Tindle, to be published; S. A. Klein, to be published.

[5]L. Durand, III, P. M. Fishbane, and L. M. Simmons, Jr., Phys. Rev. Letters 21, 1654 (1968), and to be published. The subscript on the D function in Eq. (10) of the Letter should be $-\alpha_i+l-2$. We take the scale factor in Eq. (10) as $M^2=m_1m_3$.

[6]Cf., for example, the discussion in E. J. Squires, Complex Angular Momentum and Particle Physics (W. A. Benjamin, Inc., New York, 1963).

[7]If the a_j are given, (2) becomes an nth order polynomial equation for $\bar{\alpha}_n$. The root which approaches $\bar{\alpha}_0(0)$ for $s\to 0$ can be obtained by iteration. The remaining roots do not correspond to physical Regge trajectories. The dynamical significance of the a_j, if any, is obscure.

[8]For example, the validity of the model with linear trajectories proposed by G. Domokos and P. Suranyi, Phys. Letters 24B, 293 (1967), is questionable.

[9]It follows from the results of Ref. 5 that factorization of the Regge residues restricts the possibilities to

two cases: There can be a single Lorentz pole at $s=0$ or an infinite number of poles. Any finite number $m>1$ leads to nonfactorizable Regge residues β_n for $n>m$.

[10]The little group of the Poincare group for unequal-mass scattering at $s=0$ is $E(2)$. Since this is a symmetry group of the scattering amplitude, it might be supposed that the parent-daughter sequence of Regge poles would classify in this case as a single irreducible representation of $E(2)$. This is not so. The representation coefficients of $E(2)$ are Bessel functions (cf. J. F. Boyce, R. Delbourgo, A. Salam, and J. Strathdee, to be published). No single irreducible representation can yield the expected t^α behavior of $A(s,t,u)$ for $s=0, t\rightarrow\infty$. The result obtained from a single irreducible representation of $E(2)$ is also singular in the limits $m_1-m_2\rightarrow 0, m_3-m_4\rightarrow 0$.

[11]J. B. Bronzan and C. E. Jones, Phys. Rev. Letters 21, 564 (1968).

ERRATA

SUPERCONDUCTIVITY IN MULTIPLE PAIR-BREAKING REGIMES. R. P. Guertin, W. E. Masker, T. W. Mihalisin, R. P. Groff, and R. D. Parks [Phys. Rev. Letters 20, 387 (1968)].

In line 38 of the second column on page 388, the factor 0.006 should be changes to 0.003. The same change should be made in Fig. 1 and the caption for Fig. 1.

TRAJECTORIES IN VENEZIANO'S MODEL. Ralph Roskies [Phys. Rev. Letters 21, 1851 (1968)].

In condition (A) of the theorem, replace $\ln S$ by $\ln s$. The first term of the right side of Eq. (22) should be $2\nu Ks/(\ln s)^{\nu+1}$, and the right-hand side of (25) should read

$$\exp[\tfrac{1}{2}\pi s/(\ln s)]^{\nu}\times\text{lower order terms.}$$

These changes do not affect the argument at all.

I wish to thank Professor B. Desai for pointing out the error in (25).

COULOMB DISTORTION EFFECTS IN LARGE-ANGLE $M1$ ELECTROEXCITATION. B. T. Chertok and W. T. K. Johnson [Phys. Rev. Letters 22, 67 (1969)].

A paper by Dieter Dreschel [Nucl. Phys. A113, 665 (1968)] was brought to our attention after submission of this Letter for publication. Dreschel has given an excellent quantitative evaluation of the model independency of Coulomb corrections in $M1$ electroexcitation and has demonstrated the inadequacy of the transition radius, R_{tr}, as a unique (experimentally determined) quantity for transverse transitions. Preliminary calculations of the model dependency of the $M1$ radiative widths in our work indicate, e.g., a possible additional uncertainty of 5-7 % in Γ_0 for the 11.42-MeV, 1^+ level of ^{28}Si.

The end of the last sentence of paragraph 6 should read, "$\cdots$ versus an experimental value for the ratio of the two shape factors, $A=a^-(^{12}\mathrm{B})/a^+(^{12}\mathrm{N})$, $A=(1.07\pm0.24)\%$ per MeV."