toproduction can be estimated by assuming the photoproduction cross section to be the sum of two contributions, the first being the ρ-dominance contribution, with the ratio of natural- to unnatural-parity exchanges determined by the model as

$$\frac{(\sigma_\perp^{\,+}+\sigma_\perp^{\,-})_\rho}{(\sigma_\parallel^{\,+}+\sigma_\parallel^{\,-})_\rho}=\frac{1+\rho_{1\,-1}/\rho_{11}}{1-\rho_{1\,-1}/\rho_{11}}, \tag{10}$$

and the second contribution being an ad hoc term contributing only to $\sigma_\perp^{\,+}+\sigma_\perp^{\,-}$. These assumptions give an upper limit to the fraction of pions photoproduced via the ρ-dominance terms:

$$F=[1-A(\pi^+ +\pi^-)]/(1-\rho_{1\,-1}/\rho_{11}). \tag{11}$$

Using the data shown in the figure, this upper limit is $(30\pm6)\%$, $(39\pm7)\%$, and $(60\pm21)\%$ at $-t=0.2$, 0.4, and 0.6 GeV^2, respectively. Thus, the present data imply that the original success of the ρ-dominance model in fitting the photoproduction differential cross section[1] must be regarded as fortuitous and that the unknown second term must have a t dependence similar to the ρ-dominance term; furthermore, the value of γ_ρ^{-2} must be roughly $\frac{1}{2}$ (or less) that used previously.[10]

We would like to thank the 4-GeV/c Collaboration[7] for the use of the combined data for this analysis.

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[1]C. Iso and H. Yoshii, Ann. Phys. (N.Y.) 47, 424 (1968); A. Dar, V. F. Weisskopf, C. A. Levinson, and H. J. Lipkin, Phys. Rev. Letters 20, 1261 (1968); M. Krammer, Phys. Letters 26B, 633 (1968), and 27B, 260(E) (1968); R. Diebold and J. A. Poirier, Phys. Rev. Letters 20, 1532 (1968).

[2]Chr. Geweniger et al., quoted in B. Richter, in Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, Austria, September, 1968 (CERN Scientific Information Service, Geneva, Switzerland, 1968).

[3]Chr. Geweniger et al., Phys. Letters 28B, 155 (1968).

[4]Z. Bar-Yam et al., quoted in Richter, Ref. 2.

[5]P. Heide et al., Phys. Rev. Letters 21, 248 (1968); Z. Bar-Yam et al., Phys. Rev. Letters 19, 40 (1967).

[6]See M. Krammer and Dieter Schildknecht, Nucl. Phys. B7, 583 (1968), for a derivation of similar formula.

[7]Notre Dame-Purdue-Stanford Linear Accelerator Center Collaboration, "Compilation of π^-p Data at 4 GeV/c" (to be published).

[8]In this momentum-transfer region R is considerably less than 1 and it is the π^+ asymmetry which dominates $A(\pi^+ +\pi^-)$; even if one were to ignore the experimental values and arbitrarily set A^- equal to the limit -1, there would still be a large discrepancy at $|t|=0.2$ and 0.4 (GeV/c)^2 where $A(\pi^+ +\pi^-)$ calculated in this way would be 0.19 ± 0.08 and 0.23 ± 0.09, respectively.

[9]P. Stichel, Z. Physik 180, 170 (1964); J. P. Ader et al., Nuovo Cimento 56A, 952 (1968).

[10]This would imply $\gamma_\rho^2/4\pi\geq1$, in agreement with the preliminary values reported by the Cornell and Stanford Linear Accelerator Center ρ-photoproduction groups, Fourteenth International Conference on High Energy Physics, Vienna, Austria, September, 1968 (unpublished).

EXTENSION OF THE VENEZIANO FORM TO N-PARTICLE AMPLITUDES

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The extension of Veneziano's form $V(s,t)$ to the N-particle amplitude is given.

We give here a form $V^{(N)}(s_1,s_2,\cdots)$ which is the extension to the N-particle amplitude of Veneziano's form $V^{(4)}(s,t)$ for the four-particle amplitude.[1,2] [The arguments s_i are invariant masses squared of various groups (so called "channels" i) of the N particles.] Like $V^{(4)}(s,t)$, the form $V^{(N)}(s_i)$ has an infinite number of poles in each s_i, corresponding to linear Regge trajectories of resonances, and it has Regge asymptotic behavior for large s_i; it has no cuts (hence does not satisfy unitarity), and the Regge behavior is linked with duality.

The choice of the set of the arguments s_i of $V^{(N)}(s_i)$ is guided by duality. Suppose that $V^{(N)}(s_i)$ has a pole term $\sim1/\prod_i{}'(m_i-\alpha_i)$ where each m_i is an integer, $\alpha_i(s_i)$ is the leading trajectory in channel i, and the product is over the $N-3$ internal lines of a particular N-particle tree diagram, such as in the first column of Fig. 1. Then duality implies that $V^{(N)}$ also has pole terms corresponding to other tree diagrams. A minimal set consists of all tree diagrams which are derivable

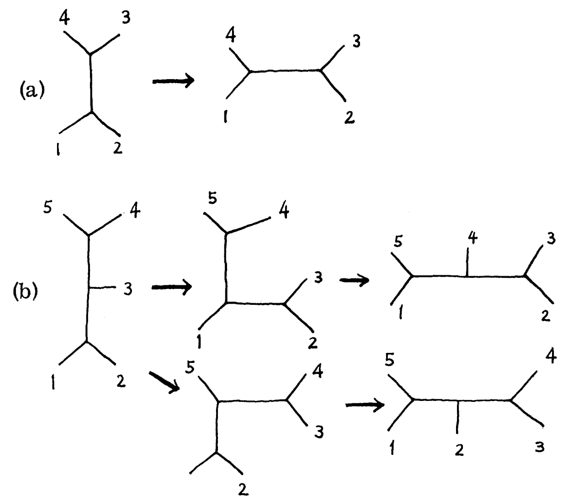

FIG. 1. Tree diagrams and duality.

from the original one by repeated replacement of an internal line by one lying at right angles. E.g., for $N=4$, it means the replacement of the diagram in which particles (12)[≡(34)] resonate by the one in which (23) resonate [see Fig. 1(a)]; similarly, for $N=5$, it means the replacement of (12)(45) by (23)(45) or by (12)(34), and the last by (51)(34), etc. [see Fig. 1(b)]. These tree diagrams are just those which are planar, the external lines remaining fixed in order. [For definiteness, we take them to be serial, $1, 2, \cdots, N$, counterclockwise.][3]

The channels i of the internal lines of the planar tree diagrams (for short, the "planar channels") are composed of consecutive particles, eg., (12), (123), (2345), etc; the corresponding s_i are $s_{12}=(p_1+p_2)^2$, $s_{123}=(p_1+p_2+p_3)^2$, etc. A shorter notation for these is s_{21}, s_{31}, s_{52}, etc., i.e., the channel index nm means the group of particles $m, m+1, \cdots, n$. A complete set of planar channels is then

$$i=nm, \text{ with } N>n>m\geq 1. \tag{1}$$

The number of planar channels is $\frac{1}{2}N(N-3)$; this exceeds the number of independent N-body scalars, i.e., $3N-10$, by $\frac{1}{2}(N-4)(N-5)$. Thus, except for $N=4$ and $N=5$, the planar s_{nm} are not independent.[4] This merely means that $V^{(N)}(s_{nm})$ equals the physical amplitude only when the $\frac{1}{2}(N-4)(N-5)$ relations (due to the four-dimensionality of space) between the s_{nm} are satisfied.

So, we require that $V^{(N)}(s_i)$ have poles at $\alpha_i=n=$ nonnegative integer, where $\alpha_i(s_i)$ is the leading trajectory in the planar channel i. This is fulfilled by the integral representation

$$V^{(N)}(\alpha)=\int_0\cdots\int_0 \prod_i \frac{dx_i}{x_i^{1+\alpha_i}}\rho(x), \tag{2}$$

where the product is over all planar channels $i=nm$, Eq. (1). [Like $V^{(4)}$, $V^{(N)}(s_i)$ depend on s_i only through $\alpha_i(s_i)$, so we write $V^{(N)}(\alpha)$.] For, if we first integrate over all variables except x_a, we have

$$V^{(N)}(\alpha)=\int_0 \frac{dx_a}{x_a^{1+\alpha_a}}R_a(x_a;\alpha_1,\cdots,\alpha_{a-1},\alpha_{a+1},\cdots), \tag{3}$$

where

$$R_a(x_a)=\int_0\cdots\int_0 \prod_{i\neq a}\frac{dx_i}{x_i^{1+\alpha_i}}\rho(x),$$

and as $\alpha_a\to n$,

$$V^{(N)}(\alpha)\to\frac{r_{a,n}(\alpha_1\cdots)}{n-\alpha_a}+\text{finite}, \tag{4}$$

where $r_{a,n}$ is a coefficient of the expansion of $R_a(x_a)$ around $x_a=0$,

$$R_a(x_a)=\sum_{j=0}^{\infty} r_{a,n}x_a^{\,n},$$

i.e.,

$$r_{a,n}=\int_0\int_0\prod_{i\neq a}\frac{dx_i}{x_i^{1+\alpha_i}}\left[\frac{1}{n!}\frac{\partial^n\rho(x)}{\partial x_a^{\,n}}\right]_{x_a=0}. \tag{5}$$

The dependence of the residue $r_{a,n}$, Eq. (5), on the α_i $(i\neq a)$ must be appropriate for the angular dependence of an amplitude with angular momentum $\leq n$, because α_a was the leading trajectory. That is, $r_{a,n}$ must be a polynomial of joint degree n in those s_i of channels i which "overlap" the s_a channel, where "overlapping" means that the two channels contain some particles in common (but neither is completely contained in the other.)[5] This "angular momentum condition" is clearly satisfied in the case $n=0$ if (and only if) $\rho(x)$ vanishes unless

$$x_i=1 \text{ when } x_a=0, \qquad \text{for any } i \text{ and } a \text{ which overlap.} \tag{6}$$

For

$$r_{a,0}=\int_0 \cdots \int_0 \prod_{i\neq a} \frac{dx_i}{x_i^{1+\alpha_i}}[\rho(x)]_{x_a=0} \qquad (5')$$

will depend on α_i unless the variable x_i equals 1, wherever $[\rho(x)]_{x_a}\neq 0$.

If $\alpha_i(s_i)$ is a linear function of s_i (as in[1] $V^{(4)}$), then the constraint (6) is sufficient to satisfy the "angular momentum condition" generally, at least if $x_i \to 1$ smoothly as $x_i \to 0$. [This can be seen by expanding $x_i^{-1-\alpha_i}$ (for those i which overlap a) in Eq. (5) around $x_i=1$ and integrating by parts.] Thus we must have a constraint of the form

$$x_i = 1-\lambda\Pi' x_a + \cdots, \qquad (7)$$

where the product is over channels a which overlap channel i, and the dots represent terms of higher degree in one or more of the x_a. There is one constraint of the form (7) for each x_i, and so unless something special happens, these equations will allow only discrete values of the x_i; the integral representation (2) will then be a sum and have no poles in the σ_i. In fact, we know that $N-3$ of the x_i must remain freely variable at least in the vicinity of $x_i=0$, in order that $V^{(N)}(\alpha)$, Eq. (2), have the multipole poles corresponding to tree diagrams.

Most remarkably, it turns out that the simplest possible form of the constraint (7), namely

$$x_i = 1-\Pi' x_a \quad \text{(product over channels } a \text{ which overlap } i) \qquad (8)$$

together with the specific choice of channels (1), leaves undetermined precisely $N-3$ of the x_i. We demonstrate this by exhibiting the x_i, expressed in terms of a set of $N-3$ independent x_i, which satisfy Eq. (8). We choose an independent x_i the x_{n1} [$n1$ are the channels of a "linear" tree diagram (multiperipheral diagram) with particles 1, 2 and $(N-1), N$ at the ends]. We abbreviate

$$x_{n1}\equiv y_n. \qquad (9)$$

The dependent x_i are then

$$x_{nm}=a_{n-1,m}a_{n,m-1}/a_{n,m}a_{n-1,m-1}, \qquad N-1\geq n>m\geq 2, \qquad (10)$$

where

$$a_{n,m}\equiv 1-\prod_{q=m}^{n} y_q, \qquad (11)$$

and

$$y_1 \equiv y_{N-1}\equiv 0, \text{ so that } a_{n,1}=a_{N-1,m}=1. \qquad (12)$$

The explicit form of the constraints (8) is

$$1-x_{nm}=\Pi x_{DC}\Pi x_{BA}\Pi y_q, \qquad (8')$$

$$1-y_n=\Pi x_{DC} \qquad (8'')$$

with the limits

$$N-1\geq D\geq n+1,$$

$$n\geq C\geq m+1,$$

$$n-1\geq B\geq m,$$

$$m-1\geq A\geq 2,$$

$$n-1\geq q\geq m.$$

(When any limit is self-contradictory, such as $m-1\geq A\geq 2$ for $m=2$, omit the corresponding product.) Substitution of (10) into the right-hand side of (8′) leads after cancellations to

$$\frac{a_{n,n}a_{m-1,m-1}}{a_{n,m}a_{n-1,m-1}}\prod_{q=m}^{n-1} y_q.$$

The left-hand side is readily seen to equal this. Likewise the right-hand side of (8″) is found to be $a_{n,n}$.

The volume element of the $(N-3)$-dimensional constrained subspace of the x_i is

$$d\mathfrak{V}=\prod_{q=2}^{N-2} dy_q \Big/ \prod_{i=2}^{N-3}(1-y_iy_{i+1}) \qquad (13)$$

because this is invariant to various replacements of the independent x_i, in particular to the replacement of the x_{n1} by x_{nk} (fixed k).[6] Putting together (2), (10), and (13), we have

$$V^{(N)}(\alpha)=\int_0^1\cdots\int_0^1\prod_{q=2}^{N-2}\left[\frac{dy_q}{y_q^{1+\alpha_{q1}}(1-y_q)^{1+\alpha_{(q+1)q}}}\right]$$

$$\times\prod_{N-2\geq B>A\geq 2}\left(1-\prod_{s=A}^{B}y_s\right)^{\alpha_{(B+1)(A+1)}+\alpha_{BA}-\alpha_{(B+1)A}-\alpha_{B(A+1)}}. \tag{14}$$

We thank M. Virasoro for much stimulation and help.[7]

Note added in proof.—The conclusion of Ref. 5 is wrong. In fact, the $\frac{1}{2}(N-4)(N-5)$ relations between the planar α_{nm} means that certain combinations of them are of lower degree in momentum transfer than their apparent degree. The "angular momentum condition" which is satisfied by $V^{(N)}$ was based on the latter, i.e., a residue's apparent degree in the α_{nm}, and so is more restrictive than necessary. For example $V^{(6)}$, used as a model for a six-pion amplitude, has no ω exchange, in say the 31 channel, because the residue of this pole term contains the form $\alpha_{42}\alpha_{53}-\alpha_{52}\alpha_{43}$, which appears quadratic, and hence was excluded at a spin-one pole of $V^{(6)}$. This form is in fact only asymptotically linear in momentum transfer because of the one relation among the nine α_{nm}. However, it is easy to construct from $V^{(6)}$ an amplitude which does have ω exchange, namely $\epsilon_{\mu\nu\rho\sigma}p_1{}^\nu p_2{}^\rho p_3{}^\sigma\epsilon_{\mu\alpha\beta\gamma}p_4{}^\alpha p_5{}^\beta p_6{}^\gamma V^{(6)}\times(\alpha_{nm}-1)$. Probably this is generally true, namely that the most general Veneziano amplitude can be constructed from the basic $V^{(N)}$ which we have presented.

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[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]The extension of Veneziano's form to $N=5$ has been made by K. Bardakci and H. Ruegg, to be published, and independently by M. Virasoro, to be published. The basic formulas [Eqs. (2) and (8)] are straight forward generalizations of their formulas.

[3]A Veneziano-type form could have pole terms corresponding to further, nonplanar, tree diagrams; an example was given by Virasoro for $N=4$ (M. Virasoro, to be published). The $V^{(N)}$ which we derive, with pole terms corresponding to only planar tree diagrams, is a basic "maximally exchange-degenerate" form, from which the general amplitude can be constructed by taking linear combination of the form with permuted external lines, as in the $N=4$ case.

[4]In a space of d dimensions, the number of independent scalars of N particles equals the number of planar channels for $N\leq d+1$; when N exceeds this, there are $\frac{1}{2}(N-d)(N-d-1)$ relations between the s_{nm}. These relations are due to the dimensionality of the space.

[5]This "angular momentum condition" is not satisfied when the $\frac{1}{2}(N-4)(N-5)$ "physical" relations between the s_{nm} are imposed, since these relations are not linear. On the other hand, it was actually stated too strictly, because angular momentum only governs the behavior of the amplitude when the momenta of the particles in a channel i are rigidly rotated rather than the amplitude's dependence on the individual s_a of channels a which overlap i. These considerations cancel out.

[6]There are other choices of the set of $N-3$ independent x_i which are not equivalent to the sets x_{nk}. (The channels of these other choices correspond to the internal lines of a branching, rather than linear, tree diagram.) Of course, the volume elements will not be of the form (13); also, in general, the dependent x_i will not be rational functions of these independent x_i (simplest example: for $N=6$, take x_{21}, x_{43}, x_{65} as independent.)

[7]After this paper was written, we received a paper by H.-M. Chan, to be published, which similarly extended the five-particle Veneziano form to $N=6$ and 7, and announced that the extension to arbitrary N had been made.