AN INTEGRAL EQUATION FOR SCATTERING AMPLITUDES*†

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We outline a dynamical scheme for studying Regge behavior of scattering amplitudes. It is based on unitarity and the concept of short-range momentum-space correlations in high-energy processes.

We describe here a type of integral equation that generalizes the 1962 model of Fubini and collaborators (ABFST model)[1] to more realistic multiperipheral mechanisms such as the multi-Regge model.[2-4] The essential requirement is that interparticle correlations in the production amplitude should involve a finite number of "links" in the multiperipheral "chain". We restrict ourselves, that is to say, to a certain type of "short-range order" in momentum space, but the details of this order are not crucial.

Virtues of the equation are these: (a) It operates entirely inside the physical region, both the kernel and the inhomogeneous term being obtainable through analysis of actual reactions. (b) It is a dynamical equation in the same sense as the Schrödinger equation, rather than a consistency condition imposed by analyticity. In fact, no reference is made to analyticity. (c) Regge poles are generated, and the consistence of input and output poles provides a natural bootstrap mechanism. (d) The dynamical relation of Regge cuts to poles is illuminated. Being an expression of unitarity, the equation includes absorptive effects.

To illustrate the method, let us suppose that the most general reaction initiated by two spinless particles a and b is $a+b\to a+b+(n$ spinless particles of mass $\mu)$. If the initial and final momenta, as well as a set of $n+1$ momenta transfers, are denoted as in Fig. 1, a simple multiperipheral model for the production amplitude assigns to it the form

$$G_a(p_0,p_1)G(p_0,p_1,p_2)G(p_1,p_2,p_3)\cdots G_b(p_n,p_{n+1},p_{n+2})+\text{exchange terms}. \tag{1}$$

In a multi-Regge model with a single type of input trajectory and without dependence on "vertex angles", for example, we would have

$$G(p_{i-1},p_i,p_{i+1})=\beta(p_i^2,p_{i+1}^2)[(p_{i-1}-p_{i+1})^2]^{\alpha_{\text{in}}(p_i^2)}, \tag{2}$$

with α_{in} the "input" trajectory and β the coupling constant for an internal vertex. The factor $G_a(p_0,p_1)$ would be the coupling constant for the end vertex connecting particle a to the chain, while G_b would have the form (2) but with the external coupling constant appropriate to particle b. More complicated models might include several input trajectories and input cuts, different kinds of produced particles,

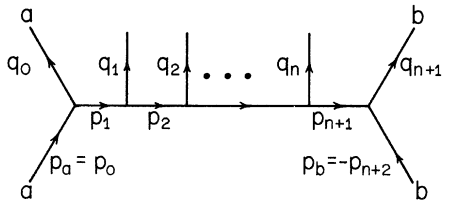

FIG. 1. The process $p_a+p_b\to q_0+q_1+\cdots q_{n+1}$.

dependence on vertex angles, etc. The associated correlation functions G would be more complicated but each would depend on a finite number of "adjacent" variables in the multiperipheral sense.

Continuing with our simple example, let us denote the absorptive part of the elastic amplitude for $a+b\to a'+b'$ by

$$\sum_{n=0}^{\infty} A_n(p_a', p_a, p_b', p_b),$$

where

$$A_n = \int d^4q_0 \cdots d^4q_{n+1}\, \delta(p_a + p_b - \sum_{i=0}^{n+1} q_i)\delta^+(q_0^2 - m_a^2)\delta^+(q_0^2 - \mu^2)\cdots\delta^+(q_{n+1}^2 - m_b^2)$$

$$\times G_a^*(p_0', p_1')G_a(p_0, p_1)G^*(p_0', p_1', p_2')G(p_0, p_1, p_2)\cdots G_b^*(p_n', p_{n+1}', p_{n+2}')$$

$$\times G_b(p_n, p_{n+1}, p_{n+2}) + \text{exchange terms}, \qquad (3)$$

with

$$p_1' = p_0' - q_0, p_2' = p_1' - q_1, \cdots, p_{n+2}' = p_{n+1}' - q_{n+1}.$$

Here δ^+ denotes the positive-energy part of the delta function. If the coupling constant $\beta(t_i, t_{i+1})$ is small except for small values of t_i and t_{i+1}, the exchange terms tend to be small. We shall here ignore them completely, although exchange can be partially included by an appropriate complication of the correlation function.

In the ABFST model $G(p_{i-1}, p_i, p_{i+1})$ contains no correlation between p_{i-1} and p_{i+1} and in fact depends only on p_i^2, being the elementary propagator for the line labeled by p_i. A recursion relation between A_{n+1} and A_n is then immediate. To accommodate our more complicated correlation we must back up one rung along the chain and undo the integration over q_0 in Formula (3). We thus define a new function B_n from which A_n is obtained as follows:

$$A_n(p_a', p_a, p_b', p_b) = \int d^4q_0\, \delta^+(q_0^2 - m_a^2)G_a^*(p_a', p_1')G_a(p_a, p_1)B_n(p_a', p_a, q_0, p_b', p_b), \qquad (4)$$

where $p_1 = p_a - q_0, p_1' = p_a' - q_0$. The quantity B_n satisfies a recursion relation of the ABFST type, namely

$$B_n(p_0', p_0, q_0; p_b', p_b) = \int d^4q_1\, \delta^+(q_1^2 - \mu^2)G^*(p_0', p_1', p_2')G(p_0, p_1, p_2)B_{n-1}(p_1', p_1, q_1; p_b', p_b) \qquad (5)$$

with $p_2 = p_1 - q_1, p_2' = p_1' - q_1$. Summing Eq. (5) from $n=1$ to ∞ and defining

$$B = \sum_{n=0}^{\infty} B_n,$$

we obtain the integral equation

$$B(p_0', p_0, q_0; p_b', p_b) = B_0(p_0', p_0, q_0; p_b', p_b)$$

$$+ \int d^4q_1\, \delta^+(q_1^2 - \mu^2)G^*(p_0', p_1', p_2')G(p_0, p_1, p_2)B(p_1', p_1, q_1; p_b', p_b), \qquad (6)$$

where B_0 corresponds to the two-particle unitarity contribution:

$$B_0(p_0', p_0, q_0; p_b', p_b) = \delta^+((p_0 + p_b - q_0)^2 - m_b^2)G_b^*(p_0', p_0' - q_0, -p_b')G_b(p_0, p_0 - q_0, -p_b). \qquad (7)$$

The complete absorptive part is obtained through the linear operation (4), i.e., integration over q_0, performed on B rather than on B_n.

A more economical notation is achieved through the change of variables, $p_0 = P_0 + \frac{1}{2}Q, p_0' = P_0 - \frac{1}{2}Q, p_1$

$= P_1 + \frac{1}{2}Q, p_1' = P_1 - \frac{1}{2}Q$, etc., plus $p_b = P_b - \frac{1}{2}Q, p_b' = P_b + \frac{1}{2}Q$. The familiar invariant variables are then

$$s = s_0 = (p_0 + p_b)^2 = (p_0' + p_b')^2 = (P_0 + P_b)^2,$$

$$s_1 = (p_1 + p_b)^2 = (p_1' + p_b')^2 = (P_1 + P_b)^2, \text{ etc.},$$

$$t_0^{(\pm)} = (P_0 \pm Q/2)^2, t_1^{(\pm)} = (P_1 \pm Q/2)^2, \text{ etc.},$$

while the actual negative momentum transfer t is

$$t = (p_a - p_a') = (p_b' - p_b)^2 = Q^2.$$

We may then rewrite our fundamental equation (6) as

$$B(P_0, P_1; P_b, Q) = B_0(P_0, P_1; P_b, Q) + \int d^4P_2\, \delta^+((P_1 - P_2)^2 - \mu^2) B(P_1, P_2; P_b, Q) H(P_0, P_1, P_2; Q), \quad (8)$$

where

$$H(P_0, P_1, P_2; Q) = G^*(P_0 - \tfrac{1}{2}Q, P_1 - \tfrac{1}{2}Q, P_2 - \tfrac{1}{2}Q) G(P_0 + \tfrac{1}{2}Q, P_1 + \tfrac{1}{2}Q, P_2 + \tfrac{1}{2}Q). \quad (9)$$

More complicated correlations can be accommodated by undoing two or more integrations in defining B. The kernel H would then depend on a correspondingly longer sequence of adjacent P's, but the form of the integral equation would be similar.

Although the driving term B_0 corresponds in the multi-Regge model to the well-known AFS cut in the angular momentum plane (the A_0 formed from B_0 is precisely this cut), the iteration of B_0 through the integral equation generates an infinite sequence of cuts that sums up to Regge poles. The underlying basis for these poles is the same as in the ABFST model: invariance of the kernel $H(P_0, P_1, P_2; Q)$ under the little group of fixed-Q simultaneous Lorentz transformations of P_0, P_1, and P_2. This invariant operation does not involve any transformation of P_b, and when s_0, s_1, and s_2 are asymptotic the invariance can be translated into the statement that the kernel depends only on the ratios s_0/s_1 and s_1/s_2, together with t, $t_0^{(\pm)}$, $t_1^{(\pm)}$, and $t_2^{(\pm)}$. Using the same reasoning as ABSFT, one may then conclude that if B is regarded as a function of the invariants s_0, s_1, $t_0^{(\pm)}$, and t, the solution of the homogeneous equation as $s_0, s_1 \to \infty$ is of the form

$$B(s_0, s_1, t_0^{(\pm)}, t_1^{(\pm)}; t) \sim s_1^{\alpha(t)} b(s_0/s_1, t_0^{(\pm)}, t_1^{(\pm)}, t_1^{(\pm)}; t), \quad (10)$$

where $\alpha(t)$ is the largest eigenvalue of the homogeneous equation. Forming A by integration then leads to

$$A(s, t_0^{(\pm)}; t) \sim s^{\alpha(t)} a(t_0^{(\pm)}; t), \quad (11)$$

the physical absorptive part to be obtained by setting $t_0^{(\pm)} = m_a^2$.

Provided the eigenvalue equation for α has a solution, there is thus a coherence leading to Regge poles in the infinite sum of cuts. The original AFS cut will not disappear but will be modified by the equation. Symbolically, if the equation is written as $B = B_0 + KB$, the solution $B = [1-K]^{-1}B_0$ contains the cut both in B_0 and in K, so there is a damping. Questions of this type we defer to a subsequent paper.

In terms of invariants the asymptotic homogeneous equation becomes

$$B(s_0, s_1; t_0^{(\pm)}, t_1^{(\pm)}; t) = g_0^2 \int \frac{ds_2}{s_1} dt_2^{(+)} dt_2^{(-)} K\left(\frac{s_0}{s_1}, \frac{s_2}{s_1}, t_0^{(\pm)}, t_1^{(\pm)}, t_2^{(\pm)}; t\right) \times B(s_1, s_2; t_1^{(\pm)}, t_2^{(\pm)}; t), \quad (12)$$

where we have explicitly extracted a factor g_0^2 to characterize the strength of an internal vertex. The eigenvalue equation corresponding to (10) is

$$b\left(\frac{s_0}{s_1}; t_0^{(\pm)}, t_1^{(\pm)}; t\right) = g_0^2 \int_0^1 dx\, x^{\alpha(t)} \int dt_2^{(+)} dt_2^{(-)} K\left(\frac{s_0}{s_1}, x, t_0^{(\pm)}, t_1^{(\pm)}; t\right) b\left(\frac{1}{x}; t_1^{(\pm)}, t_2^{(\pm)}; t\right); \quad (13)$$

the range of $t_2^{(\pm)}$ being restricted by

$$\Delta(t, t_2^{(+)}, t_2^{(-)}) = t^2 + t_2^{(+)2} + t_2^{(-)2} - 2[tt_2^{(+)} + tt_2^{(-)} + t_2^{(+)}t_2^{(-)}] < 0.$$

Now, for large s_0/s_1 and small $s_2/s_1 = x$ the multi-Regge model has the kernel behavior

$$K \sim \left(\frac{s_0}{s_1}\right)^{\alpha_{in}(t_1^{(+)}) + \alpha_{in}(t_1^{(-)})} \frac{F(t_1^{(\pm)}, t_2^{(\pm)}; t)}{[-\Delta(t, t_2^{(+)}, t_2^{(-)})]^{1/2}}. \tag{14}$$

In the weak-coupling limit, we may employ this form throughout (13) and deduce

$$b = \left(\frac{s_0}{s_1}\right)^{\alpha_{in}(t_1^{(+)}) + \alpha_{in}(t_1^{(-)})} \beta(t_1^{(\pm)}, t), \tag{15}$$

with

$$\beta(t_1^{(\pm)}, t) = \frac{g_0^2}{\alpha(t) - [\alpha_{in}(t_1^{(+)}) + \alpha_{in}(t_1^{(-)}) - 1]} \int dt_2^{(+)} dt_2^{(-)} \frac{F(t_1^{(\pm)}, t_2^{(\pm)}, t)\beta(t_2^{(\pm)}, t)}{[-\Delta(t, t_2^{(+)}, t_2^{(-)})]^{1/2}}. \tag{16}$$

Assuming F to be factorizable,

$$F(t_1^{(\pm)}, t_2^{(\pm)}, t) = F_1(t_1^{(\pm)}, t) F_2(t_2^{(\pm)}, t), \tag{17}$$

we find the eigenvalue equation

$$1 = g_0^2 \int \frac{dt^{(+)} dt^{(-)}}{[-\Delta]^{1/2}} \frac{F_2(t^{(\pm)}, t) F_1(t^{(\pm)}, t)}{\alpha(t) - [\alpha_{in}(t^{(+)}) + \alpha_{in}(t^{(-)}) - 1]}. \tag{18}$$

The corresponding Chew-Pignotti model[5] may be obtained immediately by assuming $\alpha_{in}(t)$ to be independent of t, i.e., $\alpha_{in}(t^{\pm}) \to \bar{\alpha}_{in}$. Then

$$\alpha = 2\bar{\alpha}_{in} - 1 + g^2(t), \tag{19}$$

where

$$g^2(t) = g_0^2 \int \frac{dt^{(+)} dt^{(-)}}{[-\Delta]^{1/2}} F_2(t^{(\pm)}, t) F_1(t^{(\pm)}, t). \tag{20}$$

We close with some comments on the possibility of a self-consistent weakly coupled Pomeranchuk trajectory. If we write

$$\alpha(t) = 1 - g_0^2 X(t), \tag{21}$$

and set $\alpha_{in}(t) = \alpha(t)$, then Eq. (18) becomes

$$1 = \int \frac{dt^{(+)} dt^{(-)}}{[-\Delta]^{1/2}} \frac{F_1(t^{(\pm)}, t) F_2(t^{(\pm)}, t)}{X(t^{(-)}) + X(t^{(-)}) - X(t)}, \tag{22}$$

independent of g_0^2. Assuming Eq. (22) to have a solution, it is tempting to infer a connection between the two properties $\alpha \approx 1$ and $\alpha' \approx 0$, both properties following from the smallness of g_0^2. Unfortunately, in a more realistic model, with additional input trajectories lying below the Pomeranchuk and strongly coupled, such a simple inference may not be possible. The lower lying input trajectories may play an important role in determining the properties of the output Pomeranchuk. The above example, nonetheless, illustrates the bootstrap potentialities of our equation.

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†Since this work was completed, we have received a preprint by I. G. Halliday and L. M. Saunders containing very similar ideas.

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IMPORTANCE OF THE $\eta\pi$ DECAY MODE IN THE SPLIT A_2 DECAY*

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If SU(3) symmetry is assumed and singlet-octet mixing is neglected, (a) the $\eta\pi$ decay mode is forbidden for all negative CP states, (b) $\eta\pi$ cannot appear in decays where the $K\bar{K}$ mode is absent, (c) the $\eta\pi/K\bar{K}$ branching ratio is the same for all isovector octet bosons allowed to decay into $\rho\pi$. Violation of these predictions would suggest serious re-examination of other SU(3) decay predictions. Prediction (b) seems to be violated by preliminary data. If there are no violations, and $\eta\pi$ and $\rho\pi$ are both allowed, the only possible classification for either A_2 peak with $J<4$ is 2^{++}.

The $\eta\pi$ decay mode of the lower half of the split A_2, the $A_2^L(1270)$, should be suppressed if it has $J^{PC}=1^{-+}$ as suggested by Gell-Mann and Zweig.[1] An SU(3) selection rule[2] forbids the decay into two octet pseudoscalar mesons for any boson classified in an SU(3) multiplet whose neutral nonstrange members are odd under CP. This selection rules applies to a 1^{-+} state but not to a 2^{++} state. If the lower peak is 1^{-+} and the upper is 2^{++}, as suggested by the Gell-Mann–Zweig model, the $\eta\pi$ mode should be suppressed in the lower peak relative to the upper peak. This asymmetry does not seem to be observed experimentally.[3,4]

Severe restrictions on the $\eta\pi/K\bar{K}$ branching ratio are predicted by SU(3) symmetry also for the cases where these decays are allowed, if the η is assumed to be a pure octet state. The decays of octets with even C and even P are described by a D-type coupling. All such decays are predicted to have the same $\eta\pi/K\bar{K}$ branching ratio as the 2^{++} state. If C and P are both odd, the $K\bar{K}$ mode is allowed (for the neutral case it is K_1K_2), but $\eta\pi$ and isovector $\rho\pi$ are forbidden. There is no case where $\eta\pi$ is allowed and $K\bar{K}$ is forbidden.

If a state is found which is observed to decay into $\eta\pi$, but not into $K\bar{K}$, something very peculiar and very interesting may be happening. SU(3) may be badly broken; then all previous analyses of two-body decays using SU(3) are open to question. The singlet component of the η may be dominant; then all previous estimates of η decays and decays into η's which do not take the singlet into account are suspect. There is also the possibility of peculiar interference and mixing effects which might be destructive for the $K\bar{K}$ mode in a particular experiment.[5]

For an isovector state which decays into $\rho\pi$, C is even, and the $\eta\pi/K\bar{K}$ ratio is uniquely determined, if these decays are allowed at all. The $\eta\pi/K\bar{K}$ ratio, obtained from SU(3), is equal to $\frac{2}{3}$ multiplied by the appropriate phase-space factors. Thus any variation in this branching ratio indicates either contamination by an additional odd-C state which does not go into $\rho\pi$ or one of the peculiar effects mentioned above.

Preliminary experimental results indicate that the $\eta\pi$ decay is about equal in the two A_2 peaks[3] and is symmetric in the data which do not resolve the splitting.[4] There are indications[3] of an asymmetry in the $K\bar{K}$ decay and a possible absence in the lower peak. Further experimental data on the $\eta\pi/K\bar{K}$ branching ratio in the A_2 decay is therefore of very great interest, both for charged