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MOBILITY OF THE ELECTRON BUBBLE IN SUPERFLUID HELIUM

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The phonon-limited mobility of the negative ion in He II is calculated; the results are in excellent agreement with recent experiments of Schwarz and Stark.

A slowly moving electron in superfluid helium creates around itself a large spherical bubble of radius $a \approx 15.4$ Å at zero pressure.[1] At temperatures $\lesssim 0.6°$K the drift velocity $\vec{v}_D$ of this ion in an applied uniform electric field $\vec{E}$ is limited by collisions with thermal phonons. Recent measurements[2] suggest that in the temperature range $0.3°\text{K} \leqslant T \leqslant 0.5°\text{K}$ the phonon-ion cross section for thermal phonons is inversely proportional to the phonon wave number. In this Letter we first derive, by calculating the rate of momentum transfer to the phonons, an exact expression for the ion mobility in terms of the cross section for scattering of phonons by a bubble at rest. We then calculate this cross section using a hydrodynamic approach; the required $\sim k^{-1}$ behavior of the cross section is found to arise from a prominent s-wave resonance, rather than as conjectured by Schwarz and Stark,[2] from "inelastic scattering." Finally we compare our calculations with experiment.[2]

The effective mass of the bubble is large, on the order of $(100\text{-}200)m_4$, where m_4 is the He^4 atomic mass, and thus the velocity of the ion is negligibly changed when it is struck by a long-wavelength phonon. The rate at which phonons are scattered from wave number $\vec{k}$ to wave number $\vec{k}'$ by a heavy bubble moving with steady velocity $\vec{v}_D$ is given by $n_{\vec{k}}(1+n_{\vec{k}})\Gamma_{\vec{v}_D}(\vec{k}\to\vec{k}')$, where $\Gamma_{\vec{v}_D}(\vec{k}\to\vec{k}')$ is the transition rate, and

$$n_{\vec{k}} = [e^{\hbar sk/KT} - 1]^{-1} \qquad (1)$$

is the equilibrium phonon distribution function; s is the phonon velocity. Deviations of the phonons from equilibrium that are induced by the bubble can be neglected. The net rate at which phonon momentum $\vec{P}_{\text{ph}}$ is changed by phonon-bubble collisions is

$$\frac{d\vec{P}_{\text{ph}}}{dt} = \sum_{\vec{k}\vec{k}'} \hbar(\vec{k}'-\vec{k}) n_{\vec{k}}(1+n_{\vec{k}'})\Gamma_{\vec{v}_D}(\vec{k}\to\vec{k}'). \qquad (2)$$

We are interested in terms of order $\vec{v}_D$ on the right-hand side of (2); the $\vec{v}_D$ dependence comes both from the intrinsic velocity dependence of $\Gamma_{\vec{v}_D}$ and from the k′ which contribute to (2) being Doppler shifted in magnitude from k. However, we note that were the phonons to be in equilibrium with the moving bubble, that is, were $n_{\vec{k}}$ replaced by

$$\bar{n}_{\vec{k}} = [\exp\hbar(sk - \vec{k}\cdot\vec{v}_D)/KT - 1]^{-1}, \qquad (3)$$

then $d\vec{P}_{\text{ph}}/dt$ would vanish. Thus we may write

$$\frac{d\vec{P}_{\text{ph}}}{dt} = \sum_{\vec{k}\vec{k}'} \hbar(\vec{k}'-\vec{k}) \times [n_{\vec{k}}(1+n_{\vec{k}'}) - \bar{n}_{\vec{k}}(1+\bar{n}_{\vec{k}'})]\Gamma_{\vec{v}_D}(\vec{k}\to\vec{k}'). \qquad (4)$$

The term in square brackets is explicitly of order $\vec{v}_D$, and so $\Gamma\vec{v}_D$ may be replaced by Γ_0 and the Doppler shift of k' neglected. Since $\Gamma_0(\vec{k}\to\vec{k}') = \Gamma_0(\vec{k}'\to\vec{k})$, the term in square brackets reduces to $n_{\vec{k}}-n_{\vec{k}} = \vec{k}\cdot\vec{v}_D\partial n_{\vec{k}}/\partial sk$ to first order in $\vec{v}_D$. Using isotropy we find, in unit volume,

$$\frac{d\vec{P}_{\rm ph}}{dt} = -\tfrac{1}{3}\hbar\vec{v}_D\sum_{\vec{k}} k^2\frac{\partial n_{\vec{k}}}{\partial k}\sigma_T(k), \qquad (5)$$

where

$$\sigma_T(k) = \int d\Omega(1-\cos\theta)\sigma(k,\theta) \qquad (6)$$

is the "momentum-transfer" cross section, and

$$\sigma(k,\theta) = (1/s)\textstyle\sum_{\vec{k}'}\Gamma_0(\vec{k}\to\vec{k}')\delta(\cos\theta-\hat{k}\cdot\hat{k}') \qquad (7)$$

is the differential cross section for scattering through angle θ a phonon of wave number k from a bubble at rest. Equating the left-hand side of (5) to $e\vec{E}$ we arrive at the following expression for the bubble mobility μ:

$$\frac{e}{\mu} = -\frac{\hbar}{6\pi^2}\int_0^\infty dk\, k^4\frac{\partial n_k}{\partial k}\sigma_T(k). \qquad (8)$$

Note that for σ_T independent of k, (8) reduces to $e/\mu = \rho_n s\sigma_T$, where $\rho_n \sim T^4$ is the phonon normal mass density. Equation (8) differs from the results of previous ion-mobility calculations,[3] since these calculations did not consistently account for the Doppler shift of k' and the intrinsic $\vec{v}_D$ dependence of the scattering rate. Proceeding via Eq. (4) we have bypassed these problems.

To calculate $\sigma(k,\theta)$ we use the fact that because of the relatively enormous bubble radius, a phonon-ion collision can be adequately described by the scattering of a classical sound wave by an elastic sphere of radius a at rest. The shape change of the bubble is specified by its radius change δa as a function of angle, and its dynamics by the "elastic constants" $\delta a_l/\delta p_l$ which give the $\delta a = \delta a_l Y_{lm}$ due to a pressure at the surface $\delta P = \delta p_l Y_{lm}$. Writing $x = ka$, and $\gamma_l = -(\rho_4 s^2/a)\delta a_l/\delta p_l$, where ρ_4 is the He^4 mass density, one finds by a straightforward calculation the scattering amplitude of the lth partial wave[4]

$$f_l(k) = i\frac{j_l'(x)+\gamma_l x j_l(x)}{h_l'(x)+\gamma_l x h_l(x)}, \qquad (9)$$

in terms of which

$$\sigma(k,\theta) = k^{-2}\left|\sum_{l=0}^{\infty}(2l+1)P_l(\cos\theta)f_l(k)\right|^2.$$

In (9) j_l and h_l are the spherical Bessel and Hankel functions and the prime denotes the derivative.

We turn now to the evaluation of the spring constants γ_l. All but γ_1 may be taken to be static derivatives since phonon frequencies are much less than the excitation energies of the electron in the bubble. For s waves in particular, $\delta a_0/\delta p_0$ is just $\partial a/\partial P$, the derivative of the equilibrium bubble radius with respect to pressure. From the measurements of Ref. 1 we find $\gamma_0 \approx 4.26$. On the other hand, for p waves, an infinitesimal $l=1$ displacement of the surface of the sphere corresponds to a uniform translation, for which there is no restoring force; rather, we have Newton's equation $M_B\ddot{a}_1 = -\frac{1}{3}4\pi a^2\delta p_1$, where M_B is the mass within (not the effective mass of) the bubble. Thus $\gamma_1 = -4\pi^3\rho_4/3M_B x^2$, and taking the limit $M_B \ll \frac{1}{3}4\pi a^3\rho_4$ we have $f_1 = ij_1(x)/h_1(x)$. For $ka \ll 1$ only s and p waves contribute to σ and one finds the k^4 behavior characteristic of Rayleigh scattering, $\sigma(k,\theta) = k^4 a_6[\frac{1}{3}-\gamma_0+\cos\theta]^2$, a result which agrees with a deformation-potential calculation[5] in which the effective mass of the bubble is taken to be the hydrodynamic value $M^* = \frac{1}{3}2\pi a^3\rho_4$. Figure 1 contains a plot for $ka \leqslant 4.5$ of the contri-

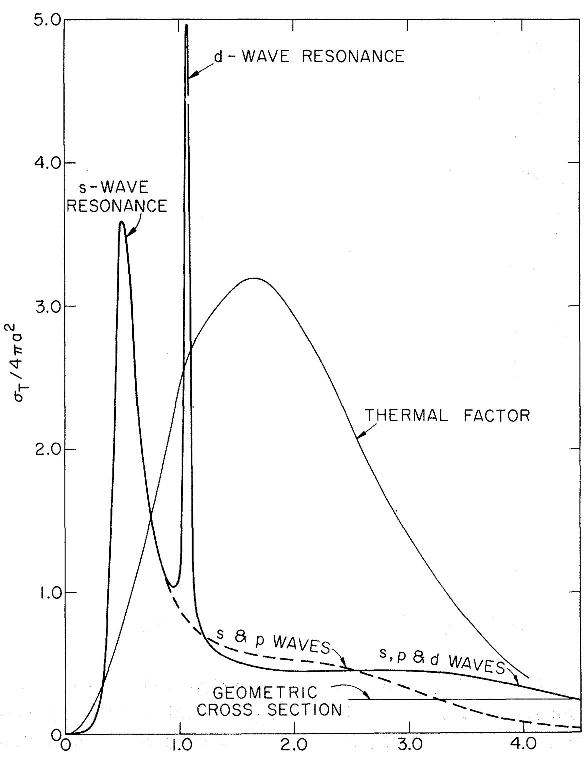

FIG. 1. The momentum-transfer cross section $\sigma_T(k)$ in units of $4\pi a^2$ for s and p waves only and for s, p, and d waves. Also shown, in arbitrary units, is the thermal factor in Eq. (8), $-k^4\partial n_k/\partial k$, for $T = 0.5$°K.

bution to $\sigma_T(k)$ arising from s and p waves alone. Note the s-wave resonance at $k \approx \gamma_0^{-1/2}/a \approx 1/2a$, due to the breathing mode of the bubble. In Fig. 2 we plot the mobility calculated using only the s- and p-wave contribution to σ_T. We shall discuss this result below.

d-wave scattering also contributes significantly to μ. Unlike γ_0, however, γ_2 cannot be determined directly from experiment. To estimate γ_2 we use the calculation of Eq. (3.10), Ref. 4, taking $a = 15.4$ Å and the effective well depth of the bubble V_0 equal to 0.6 eV, as recently measured in Ref. 1; the result is $\gamma_2 = 2.22$. The d-wave scattering has a sharp resonance at $k \approx 3/(1+3\gamma_2)^{1/2}a$, arising from quadrupole oscillations of the bubble. For $ka \lesssim 4$ the contribution to σ_T from partial waves with $l \geq 3$ affects μ by only a few percent; at higher k these partial waves ensure that σ_T attains its geometric value πa^2. We may include the effects of these partial waves to a certain extent by taking in our calculations $\sigma_T = \pi a^2$ for all $k \geq 4.4$, the value at which the s-, p-, and d-wave contribution falls to the geometric value.

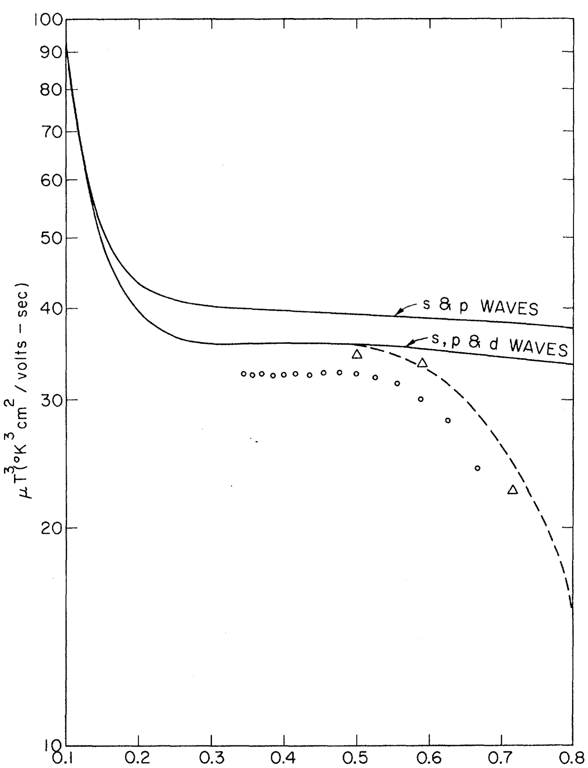

FIG. 2. Calculated phonon-limited mobilities; the upper solid curve includes only s- and p-wave scattering, while the lower solid curve includes d waves (and high partial waves, as described in the text). The dashed curve takes into account roton-bubble scattering. The circles are the data points of Schwarz and Stark, Ref. 2, and the triangles are those of Meyer and Reif, Ref. 6.

The resulting total cross section σ_T is shown in Fig. 1 and the corresponding result for the mobility is shown in Fig. 2, together with data points from Refs. 2 and 6. One striking feature of the calculated mobility is that over the temperature range 0.3-0.5°K, μT^3 is constant and equal to 35.8 cm^2 (°K)^3/V sec, which is to be compared with the experimental value[2] of 32.5 cm^2 (°K)^3 V sec. The basic reason for the constancy of μT^3 is that the effective thermal wave numbers sample the cross section in the region above the s-wave resonance where σ_T behaves essentially as k^{-1} (the thermal factor $-k^4 \partial n_k/\partial k$ peaks at $ka = 3.23T$, where T is in °K; see Fig. 1); one can see on dimensional grounds from Eq. (8) that such a cross section yields $\mu^{-1} \sim T^3$. That μT^3 is so constant is a numerical accident. At lower temperatures μ begins to rise more sharply than T^{-3}; just below 0.1°K it behaves as T^{-6}, around 0.05°K it behaves as $T^{-7.5}$, and eventually it approaches the limiting value

$$\mu^{-1} = \frac{\rho_n s}{e} 4\pi a^2 \left[\frac{2\pi a K T}{\hbar s}\right]^4 (\gamma_0^2 + 2/9) \sim T^8.$$

Towards higher temperatures, deviations from constant μT^3 behavior occur in both theory and experiment above 0.50°K. The more rapid fall of the experimental results is due to roton scattering. We can take this into account phenomenologically by adding to (8) a roton contribution, corresponding to a replacement in (8) of $s\sigma_T(k)$ by $\overline{u\sigma}$, and $\partial n_k/\partial sk$ by $\partial n_k/\partial \epsilon_k$, where $u \sim T^{1/2}$ is a mean roton-bubble relative velocity, $\bar{\sigma}$ is a mean roton-bubble momentum-transfer cross section, and ϵ_k is the roton energy. This additive contribution to e/μ is $\rho_{n,\,\mathrm{rot}}\overline{u\sigma}$, where $\rho_{n,\,\mathrm{rot}} \sim T^{\frac{1}{2}} e^{-\Delta/T}$ is the roton normal mass density and Δ is the energy of the roton minimum. The temperature behavior of this roton contribution to μ^{-1}, $\sim e^{-\Delta/T}$, agrees in the roton region with the measurements of Meyer and Reif,[6] and one can use their data to fit $\overline{u\sigma}$; taking $\bar{u} = (KT/\mu_r)^{1/2}$, where μ_r is the roton mass parameter, one finds $\bar{\sigma} \approx \frac{1}{4}\pi a^2$. The dashed extension of the calculated mobility in Fig. 2 includes this roton contribution, and is consistent with the data.

Finally we consider the sensitivity of the results to the parameters used and the question of whether the position of the d-wave resonance can be inferred from mobility measurements. Because of the breadth of the thermal factor this resonance produces no noteworthy structure in μ. We have found that the behavior of σ_T above $ka = 0.6$ is completely insensitive to a 10% variation in γ_0, and consequently, μT^3 is unchanged above 0.33°K. However, a smaller γ_0 produces a larger μT^3 at lower temperatures. We also tried $\gamma_2 = 3.92$ and found that over the range $0.3°K \leq T < 0.5°K$ μT^3 is flat like the experimental data, but its value is 10% larger than for $\gamma_2 = 2.22$. An experimental determination of μ at lower temperatures would be useful in fixing the parameters more precisely.

We are indebted to Professor T. M. Sanders for several discussions, and particularly for emphasizing the importance of the s-wave resonance, and suggesting the treatment of the $l \geq 3$ partial waves. We have also had beneficial discussions with Dr. B. Györffy and Dr. J. Lekner. David Saul has kindly assisted us with the numerical calculations.

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FREQUENCY-PULLING EFFECTS IN JOSEPHSON RADIATION*

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Using techniques developed in the quantum theory of the laser, we show that the frequency of the radiation emitted from a superconducting tunnel junction may be pulled slightly from $2eV/\hbar$ and present a physical discussion and numerical estimate of the effect.

Recent experimental[1] activity has been directed toward measuring the ratio of the electronic charge to Planck's constant. The basis[2] for these measurements is the observation that when a pair tunnels between two superconductors maintained at a potential difference V, a photon is emitted having a frequency $\omega = 2eV/\hbar$. It is the purpose of this communication to indicate a slight shift in the frequency of the emitted radiation due to cavity pulling effects. Techniques developed in treating the theory of laser oscillators[3,4] will be applied to this problem.

In our model we consider the junction to be at absolute zero and to consist of two identical thin films separated by an oxide barrier. When pairs tunnel across the barrier they excite a single mode of the electromagnetic field having an eigenfrequency Ω and a bandwidth ν/Q. In order to maintain charge neutrality pairs must be removed from one side of the barrier and added to the other by means of an external circuit.

We treat the problem as an exercise in nonequilibrium statistical mechanics in analogy with the quantum theory of the laser. The radiation-superconductor system will be coupled to reservoirs which represent the effects of a finite cavity Q in the case of the field, while in the case of the superconductors the reservoirs represent the wires (of normal metal) which are necessary in order to maintain charge neutrality. After demonstrating the frequency-pulling effect, a simple physical argument will be given which leads to the same result. It is hoped that for the reader not interested in the details of the calculation, this latter discussion will provide a satisfactory physical explanation of the frequency pulling.

We now outline a calculation[5] of the pulling effect. The state of the superconductors will be