VOLTAGE DUE TO THERMAL NOISE IN THE dc JOSEPHSON EFFECT

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The effects of thermal fluctuations on the dc Josephson effect in a junction of small capacitance are calculated using an analogy with the Brownian motion of a particle in a field of force. The results are presented in a form suitable for comparison with experiment.

At temperatures sufficiently close to the transition temperature, thermal fluctuations can disrupt the coupling of the phases of the order parameters of two superconductors separated by a thin insulating barrier. The dc Josephson[1] current thereby acquires a noise voltage with a nonzero average value. A simple kinetic theory of the mean voltage has recently been given by Ivanchenko and Zil'berman.[2] In this note we examine the problem in a slightly more detailed way, obtaining, for physical reasons discussed below, results that differ somewhat from those of Ref. 2. The problem is of current experimental interest, and a detailed comparison with experiment is in progress,[3] with satisfactory preliminary results.

Consider a Josephson junction in series with a large external resistance and battery, so that the junction is essentially being driven by a constant-current source. The equations of motion are then

$$d\theta/dt = 2eV/\hbar, \tag{1}$$

$$C\,dV/dt = I - I_1(T)\sin\theta - V/R + \tilde{L}(t). \tag{2}$$

Equation (1) is the Josephson condition relating θ, the difference in the phases of the order parameter on opposite sides of the junction, and V, the potential difference. We assume that the area of the junction is sufficiently small so that in the absence of external magnetic fields the current is uniformly distributed over the area. Equation (2) expresses the condition of conservation of charge: C is the capacitance of the junction; $I_1(T)$ is the maximum Josephson current at temperature T in the absence of noise[4]; R is the resistance of the junction, in general a function of θ and V, which for temperatures near the transition temperature, may be approximated by the (constant) resistance in the normal state[5]; and $\tilde{L}(t)$ is a fluctuating noise current. For $eV < T$, to which region we restrict ourselves, the noise is thermal,[6] so that $\langle\tilde{L}(t+\tau)\tilde{L}(t)\rangle = 2R^{-1}T \times \delta(\tau)$. The problem now has three dimensionless parameters: $\gamma \equiv \hbar I_1(T)/eT$, $x \equiv I/I_1(T)$, and $\Omega \equiv RC(2eI_1/\hbar C)^{1/2}$. This last quantity is the product of the circuit time constant $RC \equiv \eta^{-1}$ and the Josephson plasma frequency ω_J.[7]

Equations (1) and (2) are illuminated by a mechanical analogy. We rewrite these equations in the form

$$\dot{\theta} = p/M, \tag{3}$$

$$\dot{p} = dU/d\theta - \eta p - L(t), \tag{4}$$

where $p = (\hbar C/2e)V$, $M = (\hbar/2e)^2C$, $L = (\hbar/2e)\tilde{L}$, and $U = -\frac{1}{2}\gamma T(x\theta + \cos\theta)$. The problem is thus entirely equivalent to the Brownian motion of a particle of mass M in the potential U. To describe this motion we form a Fokker-Planck equation[8] for the distribution function $P(\theta, p; t)$:

$$\frac{\partial P}{\partial t} = \frac{\partial U}{\partial \theta}\frac{\partial P}{\partial p} - \frac{p}{M}\frac{\partial P}{\partial \theta} + \frac{\eta\partial}{\partial p}\left[pP + MT\frac{\partial P}{\partial p}\right]. \tag{5}$$

We restrict ourselves first to the experimentally interesting case[3] where the damping rate η is large. (A fuller discussion of this condition will be given below.) One may then use the method of Kramers[8] to reduce Eq. (5) to the Smoluchowski equation for a distribution function $\sigma(\theta, t)$ in the coordinate space alone:

$$\frac{\partial \sigma}{\partial t} = \frac{1}{\eta M}\frac{\partial}{\partial \theta}\left[\left(\frac{\partial U}{\partial \theta}\right)\sigma + T\frac{\partial \sigma}{\partial \theta}\right] \equiv -\frac{\partial w}{\partial \theta}. \tag{6}$$

We only need to consider the value of θ modulo 2π, so we may restrict θ to the interval $0 \le \theta < 2\pi$, and adjoin periodic boundary conditions to Eq. (6).

The steady-state solution of Eq. (6) must have w constant. Furthermore, if σ is normalized to unity, then w^{-1} is the average time it takes for a "particle" to diffuse one periodicity length, i.e., the average time for a phase slippage of 2π. Thus the mean voltage is given according to (1) by

$$2e\overline{V}/\hbar = 2\pi w. \tag{7}$$

The required solution of Eq. (6) is

$$\sigma(\theta) = \frac{w\eta M}{T}\frac{f(\theta)}{f(2\pi)-f(0)}\left[f(0)\int_0^\theta \frac{d\theta'}{f(\theta')} + f(2\pi)\int_\theta^{2\pi}\frac{d\theta'}{f(\theta')}\right], \tag{8}$$

where $f(\theta) = \exp[-U(\theta)/T]$. The normalization of σ then requires that

$$v \equiv \frac{\overline{V}}{I_1 R} = \frac{4\pi}{\gamma}\left\{(e^{\pi\gamma x}-1)^{-1}\left[\int_0^{2\pi} d\theta f(\theta)\right]\left[\int_0^{2\pi} d\theta' \frac{1}{f(\theta')}\right] + \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \frac{f(\theta)}{f(\theta')}\right\}^{-1}. \tag{9}$$

The curves plotted in Fig. 1 are obtained from a numerical integration of this formula. Analytic forms can be obtained in several limits:

$$v = x, \quad \gamma \to 0; \tag{10a}$$

$$= (x^2-1)^{1/2}, \quad \gamma \to \infty, x > 1;$$

$$= 0, \quad \gamma \to \infty, x < 1; \tag{10b}$$

$$\doteq 2(1-x^2)^{1/2} \exp\{-\gamma[(1-x^2)^{1/2} + x \sin^{-1}x]\} \sinh\tfrac{1}{2}\pi\gamma x, \quad \gamma \text{ large}, x < 1. \tag{10c}$$

Furthermore,

$$\lim_{x\to 0} \frac{v}{x} = [I_0(\tfrac{1}{2}\gamma)]^{-2}, \tag{10d}$$

where I_0 is the modified Bessel function.

To compare our work with that of Ref. 2, we remark that the result of the latter work corresponds to dividing the right-hand side of our Eq. (10c) by Ω. The difference has a simple physical origin. In the limit of large γ and small x, one can think of the phase slippage process as corresponding to thermal activation over a high energy barrier. The barriers of Ref. 2 and this work are identical. In Ref. 2 an attempt frequency $\omega_J/2\pi$ is introduced as an ad hoc assumption. Our calculation, while not restricted to the region of high barriers, contains in this limit the prefactor shown by Kramers[8] to be appropriate to the overdamped case, i.e., an attempt frequency $\omega_J^2/(2\pi\eta) = \Omega\omega_J/2\pi$, for $x \ll 1$. (The results of Ref 2 are probably correct for the case of small damping, provided the temperature and the value of x are sufficiently small relative to the damping.)

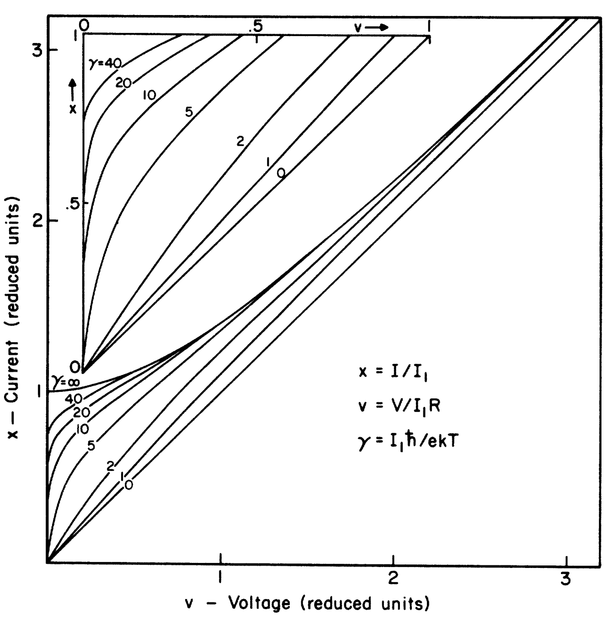

FIG. 1. Current-voltage characteristic of Josephson junction, including average noise voltage, based on Eq. (9). The inset contains an expanded version of the region $0 < x < 1$, $0 < v < 1$.

The conditions for validity of the Smoluchowski equation (6) are that the mean drift velocity of the particle be everywhere less than the thermal velocity $T^{1/2}M^{-1/2}$ and that the mean free path $T^{1/2}M^{-1/2}\eta^{-1}$ be small compared with the scale of variation in θ of the potential U. These conditions are perfectly satisfied in the limit of strong damping, $\Omega \to 0$. For Ω small but nonzero, the voltage will generally be somewhat higher than the result predicted by the Smoluchowski equation, particularly in the region $x \gtrsim 1$.

For the case of $\Omega > 1$, it becomes necessary to study the full Fokker-Planck equation (5). We hope to deal with this case in a future paper. Current-voltage curves for the Josephson junction in the absence of noise ($\gamma \to \infty$) have been discussed by McCumber and by Stewart,[9] as a function of Ω. An important qualitative feature of the curves for large Ω, which must persist to some extent when noise is included, is the occurrence of a sharp rise in voltage ("instability") at a value of x considerably smaller than unity.

It should be noted that the parameter Ω is independent of the area of the junction, depending only on the thickness of the junction, the temperature, and the materials involved. The parameter γ, by contrast, is proportional to the area of the junction.

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DIRECT OBSERVATION OF QUANTIZED MAGNETIC FLUX IN A SUPERCONDUCTING HOLLOW CYLINDER WITH AN ELECTRON INTERFEROMETER*

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Quantized magnetic flux trapped in hollow superconducting lead cylinders of extremely small dimensions (0.6 μ i.d., film thickness $\leq 0.5\ \mu$, length 100 μ) was observed in an electron interferometer. The flux was found to be quantized in integer multiples of $h/2e$ $\pm 4\%$. No influence of the size of the tube upon the fluxoid quantum has been observed.

If a superconducting hollow cylinder is cooled below the transition temperature in the presence of an axial magnetic field, magnetic flux is trapped in the cylinder after switching off the external field. This flux is quantized,[1] and past experiments[2-4] yielded the flux quantum to be $\Phi_0 = h/2e = 2.07\times10^{-7}$ G cm^2 with an accuracy of about 20%. Bardeen[5] predicted a decrease of the flux $nh/2e$ (n = integer), which is trapped in the bore, for tubes of very small inside diameter and with wall thickness of the order of the penetration depth ($\lambda_{\rm Pb} \approx 500$ Å). The cylinders used in the experiments,[2,3] however, consisted of more than 0.5-μ-thick lead or tin films on wires with diameters of 10-20 μ; their length varied between 0.6 and 9 mm. Therefore the two following questions arise: (1) Are there still principle deviations from the value $h/2e$ for one fluxoid quantum available which can be observed with a new highly sensitive measuring method? (2) Is the fluxoid quantum affected by the geometry? An experimental answer seemed to be possible with the aid of an electron interferometer, which essentially consists of an electrostatic biprism.

The biprism splits the impinging coherent beam into two electron waves, which superimpose behind the biprism giving a Fresnel diffraction pattern. If the two interfering electron beams enclose a superconducting hollow cylinder with trapped magnetic flux, a phase shift $\Delta\varphi = e\Phi/\hbar$ occurs between both waves[6,7] altering the interference figure in a characteristic way.

Taking account of the magnetic phase shift the intensity distribution of the interference pattern