MODIFIED FINITE-ENERGY SUM RULES AND THE ρ REGGE PARAMETERS*

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A new type of continuous-moment sum rule is proposed. It retains the flexibility of the the usual continuous-moment sum rule but avoids ReF in the low-energy integration. Also, it exploits the low- and intermediate-energy data more efficiently than was possible previously. We apply it to the forward pion-nucleon charge-exchange scattering to obtain $\alpha_\rho = 0.659 \pm 0.02$ and $\gamma_\rho = 0.0173 \pm 0.001$.

It was first pointed out by Igi and Matsuda,[1] and independently by Logunov, Soloviev, and Tavkhelidze,[2] that from analyticity and Regge asymptotic behavior alone it is possible to obtain information on the Regge parameters from low-energy data. As an example, they considered pion-nucleon charge exchange in the forward direction and obtained a relation between the two forward Regge ρ-exchange parameters, α_ρ and γ_ρ (the intercept and residue, respectively). In order to determine these two parameters independently, their sum rule was generalized by Dolen, Horn, and Schmid[3] and independently by Olsson[4] in two directions.

Dolen, Horn, and Schmid considered the higher-moment finite-energy sum rules (FESR). They observed that if Im$F(\nu)$ satisfies an Igi-type sum rule, ν^nIm$F(\nu)$ will satisfy it equally well. Using different moments $n = 0, 2, \cdots$, they were able to determine α_ρ and γ_ρ separately.

Olsson engaged a different sum rule, the continuous-moment sum rule (CMSR) derived from the original Gilbert dispersion relation.[5] In his sum rule, he converted the integrand Im$F(\nu)$ in the finite-energy integration into a mixture of Re$F(\nu)$ and Im$F(\nu)$. In principle these two different sum rules are as valid as the original one of Igi, but in practice both are subject to large experimental uncertainties. In one case the integrals in the higher-moment sum rule are determined almost entirely by the behavior of the function immediately below the value of the upper limit in the integration. In the other case, one has to know both the Re$F(\nu)$ and Im$F(\nu)$ in the entire range of integration. As usual, Re$F(\nu)$ is less known and often subject to large uncertainties. Furthermore, since ReF is usually calculated from the ordinary dispersion relation, high-energy information invariably creeps in through the infinite integral. Consequently, such a determination of the Regge parameters is not a bona-fide low-energy determination.[6]

In this note we propose a new type of CMSR which avoids both above-mentioned difficulties. We shall present the sum rule and discuss its merits, and then apply it to a special case, the pion-nucleon charge exchange in the forward direction for determining the ρ Regge parameters.

Consider any real analytic function $F(\nu)$ which is odd in ν and has the Regge asymptotic behavior

$$F(\nu) = F_{\text{Regge}}(\nu) \text{ for } \nu \geq \bar{\nu},$$

$$= \sum_j (\tan\tfrac{1}{2}\pi\alpha_j + i)\gamma_j(\nu)^{\alpha_j}. \qquad (1)$$

We consider the integration

$$\oint_\Gamma \left[\frac{F(\nu)}{(\bar{\nu}^2 - \nu^2)^\beta} - \frac{F_{\text{Regge}}(\nu)}{(\bar{\nu}^2 - \nu^2)^\beta}\right] d\nu = 0$$

along Γ, where Γ is the contour shown in Fig. 1 and β is any real number, $\beta < 1$. The integration along the semicircle can be ignored because $F(\nu) = F_{\text{Regge}}(\nu)$ for $\nu \geq \bar{\nu}$. We are left with

$$\int_{-\bar{\nu}}^{\bar{\nu}} \frac{F(\nu + i\epsilon)}{(\bar{\nu}^2 - \nu^2)^\beta} d\nu = \sum_j (\tan\tfrac{1}{2}\pi\alpha_j + i)\gamma_j \int_{-\bar{\nu}}^{\bar{\nu}} \frac{(\nu + i\epsilon)^{\alpha_j}}{(\bar{\nu}^2 - \nu^2)^\beta} d\nu,$$

or

$$\int_0^{\bar{\nu}} \frac{\text{Im}F(\nu)}{(\bar{\nu}^2 - \nu^2)^\beta} d\nu = \sum_i \frac{\gamma_i}{2} (\bar{\nu})^{1 + \alpha_i - 2\beta} B\left(\frac{1 + \alpha_i}{2}, 1 - \beta\right), \qquad (2)$$

where $B((1+\alpha_i)/2, 1-\beta)$ is the Euler B function. For $\beta=0$ this reduces to Igi's sum rule; for $\beta=-1$, $-2, \cdots$, it is a linear combination of the Dolen, Horn, and Schmid higher moment FESR's. It is seen easily here that the integration in the left-hand side of Eq. (2) avoids both difficulties we have just mentioned. The reason we are able to avoid an integration involving $\mathrm{Re}F(\nu)$ in the left-hand side of Eq. (2) is by the adroit choice of the modifying function $1/(\bar{\nu}^2-\nu^2)^\beta$ which has cuts only from $-\infty$ to $-\bar{\nu}$ and $+\bar{\nu}$ to $+\infty$. Such a cut structure will change the discontinuity of $F(\nu)$ from $\mathrm{Im}F(\nu)$ to $\cos\pi\beta\,\mathrm{Im}F(\nu)+\sin\pi\beta\,\mathrm{Re}F(\nu)$ only for $\nu \geq \bar{\nu}$. Thus we, too, have both $\mathrm{Re}F(\nu)$ and $\mathrm{Im}F(\nu)$ in our formalism; however, since $\mathrm{Re}F(\nu)$ occurs only beyond $\nu \geq \bar{\nu}$, it is computable in terms of the Regge parameters.

Equation (2) can be used to determine the Regge parameters accurately by fitting the numerical values of the left-hand side computed from experimental data. As an example, we consider the case where $F(\nu)$ is the pion-nucleon charge-exchange forward-scattering amplitude.[7] We use the normalization

$$\mathrm{Im}F(\nu)=\frac{k}{4\pi}\frac{1}{2}[\sigma_{\pi^-p}(\nu)-\sigma_{\pi^+p}(\nu)],$$

where k and ν are the laboratory momentum and energy, respectively. Equation (2) gives

$$I_\beta=\frac{g_{\pi N}^2}{4\pi}\frac{\nu_B}{M}\frac{\bar{\nu}^{2\beta}}{(\bar{\nu}^2-\nu_B^2)^\beta}+\frac{\bar{\nu}^{2\beta}}{4\pi^2}\int_1^{\bar{\nu}}\frac{d\nu\,k[\sigma_{\pi^-p}(\nu)-\sigma_{\pi^+p}(\nu)]}{(\bar{\nu}^2-\nu^2)^\beta}=\sum_i\frac{\gamma_i}{\pi}(\bar{\nu})^{1+\alpha_i}B\left(\frac{1+\alpha_i}{2}, 1-\beta\right), \quad (3)$$

where $\nu_B=-1/2M$. (We use natural units $\hbar=c=\mu=1$.) In Eq. (3), if the right-hand side is saturated by a single ρ Regge trajectory, α_ρ and γ_ρ are just the ρ Regge intercept and residue, respectively, with their normalization given in Eq. (1).

We choose $\bar{\nu}=6.05$ BeV/c. The left-hand side of Eq. (3) can be accurately computed from the low- and intermediate-energy data.[8] In order to minimize the systematic error in our numerical integration we limit our parameters β to the range $-1 \leq \beta \leq 0.5$, and a change of variable $k=\bar{k}\sin\theta$ makes the integrand finite everywhere and the integration much easier to handle.

The 76 integrated values I_β are shown in Fig. 2. The errors on I_β are those due to the uncertainties in the pion-nucleon coupling constant and the experimental pion-nucleon total cross sections. We note here that because of the high

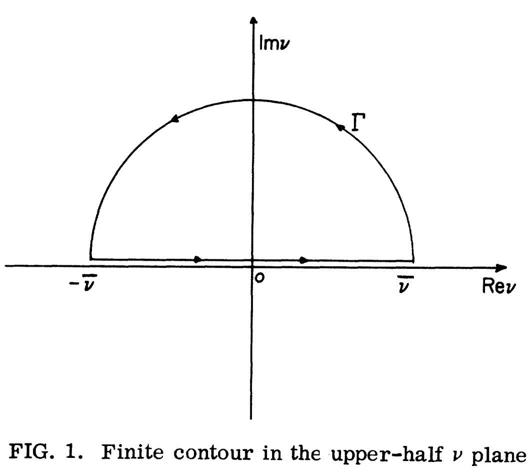

FIG. 1. Finite contour in the upper-half ν plane.

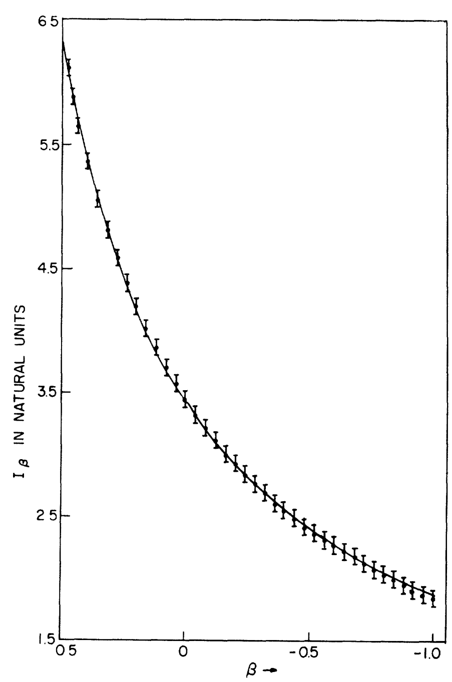

FIG. 2. Plot of I_β in natural units versus β; only alternate points are displayed to avoid overlapping. The solid curve is that calculated from the right-hand side of Eq. (3) with the parameters given in Eq. (4).

accuracy in the input data, the errors on I_β are very small ranging from 1 to 3% only.[9] We then made a search program to fit these 76 values I_β in the form of the right-hand side of Eq. (3) in terms of two parameters α_ρ and γ_ρ. Our best values for α_ρ and γ_ρ are

$$\alpha_\rho = 0.659 \pm 0.02,$$

$$\gamma_\rho = 0.0173 \pm 0.001, \qquad (4)$$

corresponding to a $\chi^2 = 33.3$. The curve shown in Fig. 2 is that calculated from the right-hand side of Eq. (3) with these values. It gives an excellent fit, taking into consideration the very small statistical errors on I_β.

We also looked for a four-parameter fit corresponding to ρ and ρ' exchanges. We found that the χ^2 value did not reduce significantly. The value $\gamma_{\rho'}$ we found is at least one to two orders of magnitude smaller than γ_ρ. We conclude that if ρ' exists, it does not contribute significantly to the pion-nucleon charge exchange, at least in the forward direction.

The value $\alpha_\rho = 0.659 \pm 0.02$ we obtain here is higher than those from previous analyses, but is not in contradiction with them. Our result is, in fact, in closer agreement with that determined by Foley et al.[10] which is very accurate. They parametrized the high-energy $\sigma_{\pi^-p} - \sigma_{\pi^+p}$ total cross section in the form B/p^c, where $p(=k$ in our notation) is the laboratory pion momentum. Their best fit is $B = 3.85 \pm 0.56$ and $c = 0.31 \pm 0.06$, corresponding to $\gamma = 0.0142 \pm 0.0021$ and $\alpha = 0.69 \pm 0.06$ in our notation. In order to see how well our values fit the high-energy cross section data, we present in Fig. 3 the experimental $\sigma_{\pi^-p} - \sigma_{\pi^+p}$ total cross sections, our predictions, and the empirical fit by Foley et al. Using our values, Eq. (4), we obtain $\chi^2 = 11.3$ for a total of eight data points; the empirical fit by Foley et al. gives $\chi^2 = 4.5$, while Olsson's value[4] gives $\chi^2 = 22.0$. Thus, not only is our method for determining the Regge parameters superior to those used in previous FESR analyses, our values also fit the high-energy data better. Finally, we made a search program for α_ρ and γ_ρ constrained to fit the 76 values I_β and the eight high-energy total cross sections simultaneously. The best values we obtained are

$$\alpha_\rho = 0.654 \pm 0.02, \qquad \gamma_\rho = 0.0176 \pm 0.001,$$

corresponding to a $\chi^2 = 42.9$ for 84 data points.

In conclusion, it is clear that our modified

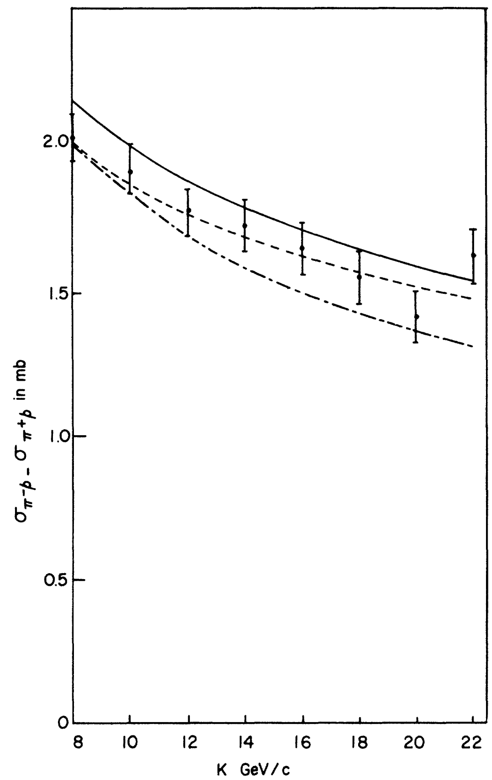

FIG. 3. $\sigma_{\pi^-p} - \sigma_{\pi^+p}$ in mb vs k in GeV/c. The solid curve is plotted using our Regge parameters Eq. (4). The dashed curve is plotted using the parametrization due to Foley et al. The dash-dotted curve represents Olsson's parametrization.

CMSR is a powerful tool for the analysis of low-energy data as it has the following advantages: (1) It avoids large uncertainties associated with ReF. (2) It is a bona-fide low-energy determination of Regge parameters as opposed to the usual CMSR determinations wherein high-energy information invariably creeps in through ReF which is calculated from ordinary dispersion relations. (3) Our formalism utilizes the low- and intermediate-energy data more efficiently because it favors higher moments.

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[7]We can, equally well, consider the pion-nucleon crossing-even amplitude, where one simply considers $\nu F^{(+)}(\nu)$ instead of $F^{(-)}(\nu)$. Such a CMSR can be used to determine the P and P' Regge parameters

[8]We use the following experimental data: The scattering lengths $a_1 = 0.192 \pm 0.004$, $a_3 = -0.096 \pm 0.002$ in natural units from G. Höhler et al., Z. Physik 180, 430 (1964). Low-energy data from V. S. Barashenkow et al., Fortschr. Physik 9, 549 (1961). Intermediate-energy data from A. A. Carter et al., Phys. Rev. 168, 1457 (1968); A. N. Diddens, E. W. Jenkins, T. F. Kycia, and K. F. Riley, Phys. Rev. Letters 10, 262 (1963); A. Citron et al., Phys. Rev. 144, 1101 (1966).

[9]There are 89 pieces of experimental data between $\nu = 0$ and $\nu = 6.05$ GeV. The integral was evaluated by asking parabolas to pass through every consecutive triplet of points. Thus $I_\beta = \sum_i c_i \sigma_i^{(-)} + a g_{\pi N}^2$. We assume that if all σ's and $g_{\pi N}^2$ are uncorrelated; then $[\sum_i (c_i^2 \delta\sigma_i^2 + a^2 \delta g_{\pi N}^2)]^{1/2}$ gives the error in I_β. We have neglected the error due to the application of Simpson's rule to the integration. Had we included systematic errors, the error in the evaluation of I_β would be larger, ranging from 3 to 8%, corresponding to a smaller χ^2 value. However, our conclusion would not be affected. See J. Orear, University of California Radiation Laboratory Report No. UCRL-8417, 1958 (unpublished).

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