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EQUATIONS OF MOTION AND THE STRUCTURE OF SINGULARITIES

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A new approach to equations of motion in general relativity is presented, in which matter is treated as singularities, of a particular kind, in the Weyl tensor. The motion of these singularities is defined in terms of their own surrounding space. The equations of motion then follow uniquely from the Einstein field equations. A concomitant result is the existence of internal structure for the singularities and equations of motion for this structure.

The purpose of this note is to point out what appears to us to be a totally new approach to equations of motion in general relativity, and to report on some unusual results that arise from this work.

The point of view which is adopted here is that matter is to be represented by singularities in the field, the field being represented by the Weyl tensor; i.e., it is a generalization of the Lorentz viewpoint to general relativity. The problem is then to give a rigorous definition of the motion of the singularity, intrinsic to its own space-time, with no reference to a background space.

The main tool for our analysis is the structure of the light cone in the neighborhood of the singularity. We thus first consider a family of null surfaces, each labeled by $u = \text{const}$, with an affine parameter r measuring "distance" on each null geodesic and two "angular" coordinates x^i labeling the geodesics. We then impose a restrictive condition which makes each of these surfaces behave like a null cone (it also fixes the origin for r), namely

$$\rho \equiv -\tfrac{1}{2} l^{\mu}{}_{;\mu} = -r^{-1} + O(r),$$

and

$$\sigma \equiv (\text{complex shear}) = O(r), \qquad (1)$$

where l^μ is the tangent vector to each geodesic. One can show from this condition (with no use of field equations) that the two-surfaces, u and r constant, possess a metric given by

$$g_{ij} = r^2 g^0{}_{ij} + O(r^3), \quad g^0{}_{ij} = g^0{}_{ij}(u, x^i).$$

We define the fundamental two-surface (F, 2-S) metric by

$$g^0{}_{ij} = \lim_{r\to 0} \frac{g_{ij}}{r^2}.$$

It can be proved that if the line $r = 0$ is a regular timelike (null, spacelike) line then the (F, 2-S) has positive (zero, negative) constant curvature.

The (F, 2-S) is most conveniently represented by conformally flat coordinates. Hence

$$dl^2 = g^0{}_{ij} dx^i dx^j = d\zeta d\bar{\zeta}/P^2, \quad \zeta = x^2 + ix^3.$$

The function $P = P(u, \zeta, \bar{\zeta})$, which will be our fundamental variable, is defined, by this construction, up to an arbitrary factor $\lambda(u)$. Later it will be shown how this factor can be normalized to 1.

If the line $r = 0$ is a singular line (i.e., the Weyl tensor becomes infinite) then the Einstein field equations yield differential equations for the determination of P. (The best known examples of this are the Robinson-Trautman type-II metrics.[1]) It is these equations which must be analyzed and from which one will extract equations of motion plus information about the structure of the singularity.

However, before we do this, it is essential for interpretive reasons to consider briefly the flat-space case of this coordinate system associated with a timelike world line.[2] Consider a timelike world line in Minkowski space, whose coordinates are given by $y^\mu = \xi^\mu(u)$, u being the proper time along the world line. The flat-space metric tensor associated with the null cones emanating from the world line can be given in the form

$$ds^2 = \left(1 - 2\frac{\dot{P}_0}{P_0} r\right) du^2 + 2du dr - \frac{r^2 d\zeta d\bar{\zeta}}{P_0{}^2}, \qquad (2)$$

where

$$P_0 = \dot{\xi}^\mu b_\mu, \quad b_\mu b^\mu = 0,$$
$$b_\mu = \frac{1}{2}\left(1+\zeta\bar{\zeta}, 1-\zeta\bar{\zeta}, \zeta+\bar{\zeta}, \frac{\zeta-\bar{\zeta}}{i}\right). \tag{3}$$

We see that the (F, 2-S) is given by $d\zeta d\bar{\zeta}/P_0^2$. (It can be easily checked that it is the unit sphere.) The important point is that from properties of P_0 one can extract all the information about the timelike line. For instance, the maximum value of $\dot{P}_0/P_0$ at each u, maximized with respect to ζ and $\bar{\zeta}$, yields

$$\max \dot{P}_0/P_0 = (\ddot{\xi}_\mu \ddot{\xi}^\mu)^{\frac{1}{2}}. \tag{4}$$

The angle ζ and $\bar{\zeta}$ which yields this maximum is the direction of the acceleration. We can thus think of a one-to-one relation between $\dot{P}_0/P_0$ and the acceleration vector $\ddot{\xi}^\mu$. This interpretation of P_0 and its derivative is basic for what follows.

As we can only here sketch the method of obtaining the equations of motion, it is best to illustrate it in a special case, namely when Eq. (1) becomes

$$\rho = -r^{-1}, \quad \sigma = 0.$$

This restriction leads to the Robinson-Trautman metrics[1] which are now summarized:

$$ds^2 = 2\left[K - \frac{\dot{P}}{P}r - \frac{M(u)}{r}\right]du^2 + 2du dr - \frac{r^2}{2}\frac{d\zeta d\bar{\zeta}}{P^2}$$

(there is a minor notational change with the factors 2 and $\frac{1}{2}$), where[3] $K \equiv \eth\bar{\eth}\log P$ is the Gaussian curvature of the (F, 2-S) $[\eth\bar{\eth} \equiv 4P^2(\partial^2/\partial\zeta\partial\bar{\zeta})]$ and $M(u)$ and $P(u, \zeta, \bar{\zeta})$ satisfy

$$\dot{M} - 3\frac{\dot{P}}{P}M = \eth\bar{\eth}K. \tag{5}$$

From this equation we will be able to obtain the time dependence of the mass M, the acceleration of the singularity, and equations of motion for the internal degrees of freedom (of the singularity) yet to be defined. In all our generalizations, i.e., to general $R_{\mu\nu} = 0$ with Eq. (1) and to Einstein-Maxwell theory, Eq. (5) becomes modified by the addition of extra terms, which are interpreted to represent the interaction of a background field (suitably modified by the presence of the singularity), both gravitational and electromagnetic, with the singularity.

We now impose the restrictive condition on the solutions, that P can be written as

$$P = P_0(1+I), \tag{6}$$

where I is a regular function on the sphere expandable in spherical harmonics starting with $l=2$ [this latter condition is not a real restriction; it arises from normalizing the multiplication factor, mentioned earlier, in the P and from the fact that any $l=1$ spherical harmonic can be included in the P_0] and P_0 is that given in Eq. (3). This condition is merely the statement that the (F, 2-S) is a deformed sphere. When Eq. (6) is substituted into Eq. (5), the resulting equation can in principle (though not in practice, due to the extreme nonlinearity) be expanded in spherical harmonics. The $l=0$ equation gives the equation for the time development of the mass M; the $l=1$ equation gives the equation for $\dot{P}_0/P_0$, i.e., the acceleration according to our interpretation, Eq. (4); and all the higher l equations give the time dependence of the different spherical harmonics of I, which are interpreted as internal degrees of freedom.

Though we have here, in principle, exact equations of motion, their analysis has been possible only by an approximation method, namely by expanding the equations in powers of I and by keeping only the first and second powers. Here we will just state some of the results; the details will be described elsewhere.

(1) Robinson-Trautman.—In this case, if no internal degrees of freedom are excited one has only the Schwarzschild solution. If internal degrees of freedom are excited, i.e., $I \neq 0$, then they exponentially decay and the solution quickly returns to Schwarzschild, emitting gravitational radiation in the process. If internal degrees of freedom with both even and odd spherical harmonics are stimulated then momentum is radiated and the singularity recoils, i.e., $\dot{P}_0/P_0 \neq 0$. We have also derived and analyzed the charged counterparts of the Robinson-Trautman metrics.

(2) General case of $R_{\mu\nu} = 0$.—Actually in these calculations, we have specialized by setting a pair of constants of integration (in linear theory the dipole moment and angular momentum) equal to zero. In this case the internal degrees of freedom and acceleration are excited by the order-of-magnitude terms in Eq. (1). If the strength of the singularity (the mass) is permitted to approach zero, i.e., if we take the test-particle limit, then $\dot{P}_0/P_0 = 0$, which is identical with $r=0$ being a geodesic in the now regular background space. The I, of course, goes to zero.

We obtain an interesting and very satisfactory result if, before passing to the test-particle limit, we constrain the internal degrees of freedom to be zero. $r=0$ is then still a geodesic, but in addition there must be neighboring geodesics with zero geodesic deviation from it – the geodesic must be part of a rigid congruence. This gives an intuitive picture of the internal degrees of freedom responding to the equation for geodesic deviation.

(3) Einstein-Maxwell theory. – In this case the analysis is many times more difficult than in the previous cases and has not been completed. Nevertheless, to lowest order in the approximation, several interesting results and one surprising result are obtained. The Lorentz force law in terms of a unique "background" is obtained, as well as electromagnetic interactions with I. The surprising result was the appearance of the radiation reaction term $\frac{2}{3}e^2(\dddot{\xi}^\mu + \ddot{\xi}^\alpha \ddot{\xi}_\alpha \dot{\xi}^\mu)$ in the force law, with no mass renormalization term. This result is preserved under the test-particle limit. It appears to us as if this is a major success of this approach. It supports speculations that a quantized version of general relativity might eliminate some of the self-energy difficulties in quantum electrodynamics.

It seems natural to interpret the singularities in Robinson-Trautman solutions as "elementary" singularities, with no interactions with external sources, while the more general solutions would represent the interaction of these "elementary" singularities with sources or a background field. The same could be said of the charged counterparts.

There appears to be a major drawback in this approach to equations of motion. The acceleration has been formally defined in terms of $\dot{P}_0/P_0$, in analogy with flat space. But no means has been presented by which, even in principle, $\dot{P}_0/P_0$ and hence the acceleration could be measured. The situation is, however, not hopeless – it appears likely that one can place a family of geodesic observers into such motion that by observing the time derivative of the Doppler shift, they would be measuring directly $\dot{P}_0/P_0$. This can be done when the world line $r=0$ is nonsingular. This point will be amplified on in the more complete version of this work.

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CORRELATION BETWEEN FISSION TRACKS AND FISSION-TYPE XENON FROM AN EXTINCT RADIOACTIVITY*

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Meteoritic whitlockite containing excess fission tracks has a large concentration of excess neutron-rich xenon isotopes which is 25 times that calculated from the track density. The isotopic spectrum is identical to that calculated previously for the Pasamonte achondrite. These results uniquely associate this spectrum with in situ fission. Identification of the fissioning nucleus as Pu^{244} gives $Pu^{244}/U^{238} \approx 1/30$ at the time of xenon retention. Neither "sudden" nor "uniform" nucleosynthetic models give consistent solutions for Pu^{244}/U^{238} and U^{235}/U^{238}.

There are two independent lines of evidence for fission products in meteorites: (1) Several meteorites contain distinct enrichments in the isotopic abundance of the neutron-rich Xe isotopes suggestive of fission.[1-3] (2) Studies on minerals from several other meteorites have revealed high densities of charged-particle tracks with properties characteristic of fission tracks rather than tracks of heavy cosmic-ray ions.[4-6]

In all cases studied the concentrations of U appear to be inadequate to account for either the observed track densities or the amounts and isotopic composition of the excess Xe in terms of spontaneous fission or induced fission. The excess fission tracks and excess neutron-rich Xe are usually attributed to the spontaneous fission of Pu^{244}