[12]J. H. Lee and J. Wilson (Imperial College, London, England), private communication.
[13]N. G. Basov et al., Dokl. Akad. Nauk SSSR 173, 538 (1967) [translation: Soviet Phys.—Doklady 12, 248 (1967)].
[14]M. Born and E. Wolf, Principles of Optics (Pergamon Press, New York, 1964).
[15]F. Zernike and B. R. A. Nijboer, contribution to La Théorie des Images Optiques (Revue d'Optique, Paris, 1949), p. 232.
[16]E. H. Linfoot and E. Wolf, J. Phys. B: Phys. Soc. (London) Proc. 69, 823 (1956).
[17]A. Maréchal, see E. H. Linfoot, Recent Advances in Optics (Clarendon Press, Oxford, 1955).
[18]L. R. Evans and C. Grey Morgan, Nature (London) 219, 712 (1968).
[19]L. R. Evans and C. Grey Morgan, Phys. Med. Biol. 14, 2 (1969).

ORDER-PARAMETER FLUCTUATIONS AND THE ONSET OF SUPERFLUIDITY IN THIN He^4 FILMS

M. Revzen*
Department of Physics, University of California, Los Angeles, California 90024
(Received 3 April 1969)

A new method for calculating the resistance of superfluids is proposed. The method is applied to derive the temperature-dependent thickness of He^4 films at which superflow ceases. The agreement with the experimental results is good.

The calculations of the electrical conductivity for thin superconductors are conveniently made[1] in the gauge where ($c = 1$)

$$\vec{E} = \partial\vec{A}/\partial t, \tag{1}$$

where $\vec{E}$ is the longitudinal electric field and $\vec{A}$ is the vector potential. This gauge is convenient because in effect, it couples the external electric field to the electric current directly.[2] In the case of superfluids an external longitudinal field is coupled to the density and hence the calculation of the conductivity density (i.e., the ratio of the current to an external field in the limit of small frequency and small wavelengths) is more complicated.[3] In the first part of this note we argue that any external longitudinal field can be made to couple to the velocity in close analogy to the electromagnetic situation. Once this is done we transcribe the calculated results[1,4,5] for the resistance of thin (two dimensional, 2D henceforth) superconductors to that of the uncharged superfluid. We then show that the calculated values are in fair agreement with various experimental results. In this connection we make two remarks: (a) In superfluids one does not measure the resistance of the superfluid as a function of temperature for a fixed film thickness (d). Rather, one observes[6] the onset of superflow for a fixed temperature as a function of thickness. (b) The meaning of 2D is different in the two cases (superfluid and superconductor) because of the different boundary conditions which are considered. Thus the superconductor's wave function (in the Landau-Ginzburg sense, LG henceforth) has a vanishing normal derivative at the surface of the film. Here, then, 2D means $d \lesssim \xi$ (ξ is the LG coherence length). This is so because for these thicknesses the order parameter is either constant or zero. For superfluids the wave function (in the Ginzburg-Pitaevskii sense, GP henceforth) is zero at least at the interface with the substrata. Here $d \lesssim \xi$ has the meaning of no superfluidity ($\psi = 0$). In this case 2D means $2\xi \lesssim d/\pi \lesssim \xi$ because for these thicknesses the wave function is either zero or $\psi \sim \sin(x/\xi)$.

We now show the transformation between the wave equation wherein a longitudinal field is coupled to the density, and the equation wherein the equivalent field is coupled to the velocity. The argument is simplest in the time-dependent Schrödinger equation.[7] Consider the many-body equation[8]

$$-(\hbar^2/2m)\nabla^2\psi + \varphi(x)\psi = i\hbar\partial\psi/\partial t. \tag{2}$$

(Introduction of an interaction term does not affect the argument.) In Eq. (2), $\varphi(x)$ is the scalar field whose gradient gives the externally applied force. Now define χ via

$$\psi(x_1, x_2, \cdots, x_N, t) = \exp[-i\sum_{i=1}^{N}\int_0^{x_i}\vec{A}(y_i, t)\cdot dy_i] \times\chi(x_1, \cdots, x_N, t). \tag{3}$$

Here $\vec{A}$ is a curl-free vector field. By direct substitution of (3) into (2) one gets[8]

$$-(\hbar^2/2m)(\nabla - iA)^2\chi = i\hbar\partial\chi/\partial t \tag{4}$$

provided we choose $\vec{A}(x,t)$ such that

$$\varphi(x) = \hbar\int^x (\partial\vec{A}/\partial t)\cdot dy. \tag{5}$$

Since $\vec{A}$ is curl free, the lower bound of this integral corresponds to an irrelevant constant phase factor. For example if the external force is[2]

$$F(x,t) = \vec{F}_0[e^{i(q\cdot r - \omega t)} + \text{c.c.}]e^{st}, \quad s \to 0^+, \tag{6}$$

then

$$A(x,t) = \frac{\vec{F}_0}{\hbar}\left[\frac{e^{i(q\cdot r-\omega t)}}{i\omega + s} + \text{c.c.}\right]e^{st}, \quad s \to 0^+. \tag{7}$$

We do not consider the boundary conditions on χ here[7] because we are interested in an extended system and, in fact, we are concerned with

$$Z_G = \mathrm{Tr}[e^{-\beta(H-\mu N+\varphi)}], \tag{8}$$

where Z_G is the grand canonical ensemble for bosons and φ is the relevant external field. H and N are the Hamiltonian and number operators, respectively, and

$$\beta = (k_\mathrm{B}T)^{-1}$$

with T the temperature and k_B Boltzmann's constant. Now Z_G can be transformed to a functional integral[9] wherein the "time"-dependent Lagrangian appears. In this formulation the transformation $\varphi \to \chi$ corresponds to a change of variables. The Jacobian of this transformation will cancel out[10] when physical quantities are calculated.[11]

We can now calculate the resistance of 2D superfluids in complete analogy to the calculations of electrical resistance.[12] In particular we consider the GP Hamiltonian. Here the calculations are identical to those of Ref. 4 where the LG Hamiltonian was used. (Recall that LG is essentially identical in form to GP.) The calculation for the excess conductivity (σ') in the "clean" limit[13] gives[4]

$$\sigma' \simeq \frac{m}{\hbar d}\exp\left[\frac{4\pi}{k_\mathrm{B}T}\left(\frac{\hbar}{m}\right)^2\rho_s d\right], \tag{9}$$

where the bulk superfluid density ρ_s is given by $\rho_s = m(|a|/b)$, with m now being the mass of a helium atom. a and b are coefficients of binary and tertiary terms in the GP equation. We now remark that an important experiment result[6,14] is that the bulk superfluid density varies with temperature as[15]

$$\rho_s(T) = \frac{B_0 T}{\xi(T)}, \quad T_\lambda \geq T \geq 1°\mathrm{K}, \tag{10}$$

with $B_0 = 2.9\times10^{-9}$ g cm^{-2} deg^{-1}. This relation will be used below as our definition of $\xi(T)$ for $T \leq T_\lambda$.

At this juncture we introduce a free parameter σ_c' into the theory. σ_c' is the value of the excess conductivity upon the attainment of which the system has an essentially normal behavior, i.e., $\sigma' \gg \sigma_c'$ implies superfluidity and $\sigma' \ll \sigma_c'$ implies normal resistance.[5] In terms of this parameter we can rewrite Eq. (9) as

$$\sigma'(d) = \sigma_c' \exp\left[\frac{d-d_c}{Q\xi(T)}\right], \tag{11}$$

where $d_c(T)$ is the equilibrium thickness for which $\sigma' = \sigma_c'$ and Q is a dimensionless constant

$$Q^{-1} = \frac{4\pi B_0}{k_\mathrm{B}}\left(\frac{\hbar}{m}\right)^2 = 4.5. \tag{12}$$

This exponential dependence[13] renders plausible the following two remarks: (a) σ_c' is not sensitive to the particular experimental method of study of the superflow. (b) All[13] the temperature dependence of the problem is in the ratio $d(T)/\xi(T)$.

Now the equilibrium value of $d(T)$ is given by[6,14]

$$d^3 = \frac{87°\mathrm{K}}{T\ln(P_0/P)}. \tag{13}$$

Here d is given in atomic layers while P_0 and P are the saturated and unsaturated vapor pressure of the helium gas.

We can now get the superflow onset thickness for all temperatures.[15] Thus since σ_c' depends only on d_c/ξ, the onset condition for a fixed T is

$$d_c/\xi = \text{const.} \tag{14}$$

(This constant turned out to be ~4.2; see Fig. 1.) This equation gives the critical thickness for various temperatures. Relation (14) is plotted in Fig. 1 together with several experimental results. The value of $\xi(T)$ is the one given by Eq. (10) for which the tabulated values[16] for $\rho_S(T)$ are used. We remark that at the higher temperatures $T \gtrsim 1.7°$K slight error in the experimental onset parameter P_0/P results in large error in the critical thickness d_c.[6] Thus the scatter of the experimental points in this region is not surprising.

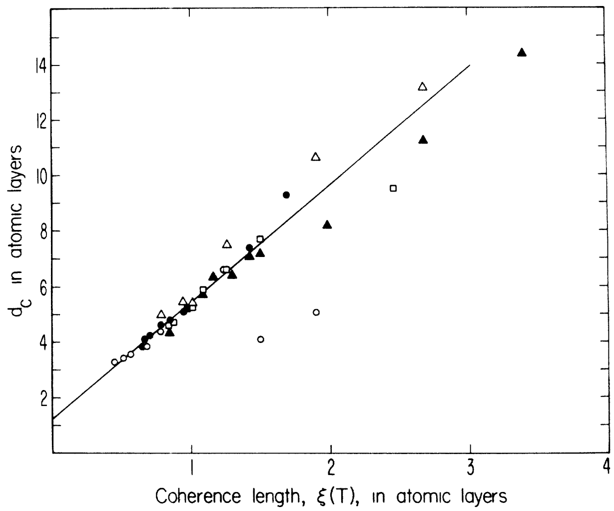

FIG. 1. Helium film thickness at the onset of superfluidity, d_c, as a function of the temperature-dependent length $\xi(T)$ as determined by various experiments. Onset of mass superflow from E. Long and L. Meyer [Phys. Rev. 85, 1030 (1952)] (solid triangles) and R. P. Henkel, G. Kukich, and J. P. Reppy [Proceedings of the Eleventh International Conference on Low Temperature Physics, St. Andrews, Scotland, 1968 (to be published)] (open diamonds). Onset determined by heat transport from E. Long and L. Meyer [Phys. Rev. 98, 1616 (1955)] (open triangles), D. F. Brewer and K. Mendelsohn [Proc. Roy. Soc. (London), Ser. A 260, 1 (1961)] (open squares), and K. Fokkens, W. K. Taconis, and R. De Bruyn Ouboter [Physica 32, 2129 (1966)] (open circles). Third-sound onset data from Kagiwada et al. (Ref. 6) (solid circles). The intercept at $\xi = 0$ is interpreted as the solid layer in the helium film (Ref. 6).

The intercept at $\xi = 0$ (i.e., $T = 0$) is interpreted as the solid He layer. The van der Waals force gives rise to pressure greater than the solidification pressure for helium for about 1.3 atomic layers.[6] This number is in good agreement with intercept of Fig. 1 which gives 1.2 atomic layers.

We conclude by noting that the results seem to confirm the intuitive idea that the fluctuation in the order parameter gives rise to the appearance of resistance in superflow. These fluctuations are the same as those which give rise to the electrical resistance in superconductor films. The present formulation seems to give a good account of the experimental result without recourse to a specific model for fluctuation.[14]

Correspondence and conversations with S. Marcelja, W. E. Masker, and R. D. Parks are gratefully acknowledged. Discussions with Professor N. Byers, Professor T. Holstein, and Professor K. Maki, and also with Mr. J. C. Fraser were very helpful. In particular, I benefitted from several discussions with Professor I. Rudnick.

---

*Permenent address: Physics Department, Technion —Israel Institute of Technology, Haifa, Israel.

[1]L. G. Aslamazov and A. I. Larkin, Fiz. Tverd. Tela 10, 1104 (1968) [translation: Soviet Phys.—Solid State 10, 875 (1968)], and Phys. Letters 26A, 238 (1968).

[2]T. Holstein, Ann. Phys. (N.Y.) 29, 410 (1964).

[3]R. Kubo, J. Phys. Soc. Japan 12, 570 (1957).

[4]S. Marčelja, Phys. Letters 28A, 180 (1968); S. Marčelja, W. E. Masker, and R. D. Parks, Phys. Rev. Letters 22, 124 (1969).

[5]L. P. Kadanoff and G. Laramore, Phys. Rev. 175, 579 (1968).

[6]R. S. Kagiwada, J. C. Fraser, I. Rudnick, and D. Bergman, Phys. Rev. Letters 22, 338 (1969).

[7]N. Byers and C. N. Yang, Phys. Rev. Letters 7, 46 (1961).

[8]For simplicity of representation, summation over the single particles is deleted. Each single-particle operator should be regarded as a sum over single-particle operators.

[9]See, for example, A. Casher, D. Lurie, and M. Revzen, J. Math. Phys. 9, 1312 (1968).

[10]That the Jacobian can be taken out of the functional integral is obvious on intuitive grounds; nonetheless, this should be regarded as a conjecture. See, however, Sec. IV of Ref. 9 and references cited therein.

[11]The lower bound phase factor [Eq. (5)] is here a constant term added to the chemical potential and thus is not physically observable.

[12]In particular the current J is related to the free energy F by $J = \delta F/\delta A$.

[13]Here and in the sequel we neglect the film-thickness dependence of the prefactor of Eq. (9). Thus, in effect, the "clean" and "dirty" limits are equivalent here.

[14]K. Maki, M. Revzen, and I. Rudnick, to be published.

[15]This expression reduces to the scaling-law result for $T \to T_\lambda$ [e.g., B. R. Josephson, Phys. Letters 21, 608 (1966)]. However, Eq. (15) remains valid at temperatures where neither ρ_s nor ξ have their asymptotic ($T \to T_\lambda$) temperature dependence.

[16]J. G. Dash and R. D. Taylor, Phys. Rev. 105, 7 (1957).