We used a more recent number for Γ reported in N. Barash-Schmidt et al., University of California Radiation Laboratory Report No. UCRL-8030 Revised, 1968 (unpublished).

[5]N_+ and N_- are the number of events with $x>0$ and $x<0$, respectively, in the Dalitz plot.

π-π PARAMETERS AND SUM RULES AT THE SYMMETRY POINT*

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We show that the current-algebra predictions of Weinberg for π-π scattering may be consistent with the sum rules proposed by Chu and Desai, who have claimed otherwise. We demonstrate that high-spin resonances neglected by Chu and Desai can make substantial contributions to the sum rules, and that inclusion of the next resonance after those used by Chu and Desai results in agreement between Weinberg's predictions and the sum rules. It is possible and likely that the remaining high-spin resonances give negligible net contribution.

Consider the π-π elastic scattering amplitudes $A^I(\nu,\cos\theta)$, where $\nu=s/4-m_\pi^2$. At the symmetry point (defined by $s=t=u$), ν and $\cos\theta$ have the values $\nu_0=-\frac{2}{3}m_\pi^2$, $\cos\theta=0$. Crossing symmetry permits us to introduce two parameters λ and λ_1 which satsify

$$\lambda=-\tfrac{1}{5}A^0(\nu_0,0)=-\tfrac{1}{2}A^2(\nu_0,0),$$

$$\lambda_1=\frac{1}{2}\frac{\partial A^0}{\partial\nu}\bigg|_{\nu_0,0}=-\frac{\partial A^2}{\partial\nu}\bigg|_{\nu_0,0}=\frac{\partial}{\partial\cos\theta}\left(\frac{A^1}{\nu}\right)_{\nu_0,0}.$$

By dispersing $A^2(\nu,\cos\theta)$ at fixed energy, Chu and Desai[1] (hereafter CD) obtain the formula[2]

$$\lambda_1=\frac{1}{\pi}\sum_{I=0}^{2}\alpha_{2I}\int_0^\infty\frac{d\nu'}{\nu'-\nu_0}\left(\frac{1}{\nu'-\nu_0}-\frac{4}{\nu'}\frac{\partial}{\partial\cos\theta'}\right)\times\mathrm{Im}A^I(\nu',\cos\theta'),\qquad(1)$$

where $\alpha_{2I}=\frac{1}{3}$, $-\frac{1}{2}$, and $\frac{1}{6}$ for $I=0$, 1, and 2, and where $\cos\theta'=1-\nu_0/\nu'$. In addition, CD assume that $A^2(\nu,\cos\theta)$ satisfies an unsubtracted dispersion relation for fixed energy, so that

$$\lambda=-\frac{1}{\pi}\int_0^\infty\frac{d\nu'}{\nu'-\nu_0}\sum_{I=0}^{2}\alpha_{2I}\mathrm{Im}A^I(\nu',\cos\theta').\qquad(2)$$

CD report having made a "semiquantitative estimate" that the integrands of Eqs. (1) and (2) are negligible for center-of-mass energies above 1500 MeV, and they use this estimate to justify neglecting the contributions of the ρ recurrence, $g(1650)$, and all higher mass contributions. They then carry out an analysis from which they conclude that λ must be small and positive, in contradiction to the current-algebra prediction of Weinberg.[3] In addition, CD conclude that λ_1 is positive but substantially smaller than the value predicted by Weinberg. However, let us explicitly evaluate the ρ, f_0, and g contributions[4] to λ and λ_1 by using the Breit-Wigner formula[5]

$$\mathrm{Im}A^{(l)I}(\nu)=\frac{(1+1/\nu)^{\frac{1}{2}}\gamma_R^{\,2}\nu^{2l+1}}{16(\nu+1)(\nu-\nu_R)^2+\gamma_R^{\,2}\nu^{2l+1}},$$

where

$$\gamma_R^{\,2}\equiv4(\nu_R+1)^2\Gamma_R^{\,2}/\nu_R^{\,2l+1}.$$

Assuming $\Gamma(\rho)=(130\pm20)$ MeV, $\Gamma(f_0)=(140\pm15)$ MeV, and that ρ and f_0 decay into two pions with unit probability, it follows that

$$\lambda(\rho)=0.26\pm0.04,$$
$$\lambda(f_0)=-0.18\mp0.02,$$
$$\lambda_1(\rho)=(0.106\pm0.014)m_\pi^{-2},$$
$$\lambda_1(f_0)=(-0.093\mp0.008)m_\pi^{-2},$$

where the ρ and f_0 contributions to the integrals in Eqs. (1) and (2) have been cut off at $E_{\mathrm{c.m.}}=1000$ and 1500 MeV, respectively.[6] The width $\Gamma(g)$ and the probability ξ of $g\to2\pi$ are less well known, so we choose to express $\lambda(g)$ and $\lambda_1(g)$ in terms of them. For $\Gamma(g)$ in the range $\Gamma(g)=(140\pm70)$ MeV $=(1.0\pm0.5)m_\pi$, we find that λ and λ_1 are proportional to $\Gamma(g)$ within 5%, with coefficients given by

$$\lambda(g) \cong 0.31\xi\Gamma(g)m_\pi^{-1},$$

$$\lambda_1(g) \cong 0.186\xi\Gamma(g)m_\pi^{-3},$$

where units of m_π are to be used for $\Gamma(g)$, and where the integrals have been cut off at $E_{\mathrm{c.m.}} = 2000$ MeV.[6] Observe that if ξ were unity and $\Gamma(g)$ were as large as $\Gamma(\rho)$, then we would have $\lambda(g) \approx \lambda(\rho)$ and $\lambda_1(g) \approx 2\lambda_1(\rho)$![7] We shall return later to a discussion of the experimental values of $\Gamma(g)$ and ξ.

For the S-wave contributions, let us consider S-wave solutions obtained by the method of Tryon[8,9] which incorporate Weinberg's predictions[3,10] that

$$\lambda_1 \cong (0.105 \pm 0.015)m_\pi^{-2}, \tag{3a}$$

$$\lambda \cong -(\lambda_1/12)m_\pi^{2} \cong -0.01, \tag{3b}$$

$$a_0 \cong -\tfrac{7}{2}a_2 \cong (7/4)\lambda_1 m_\pi, \tag{3c}$$

where a_I is the S-wave scattering length of isospin I. An interesting feature of Weinberg's predictions is that while a_0 is small, the value of a_0 together with the large value predicted for $dA^{(0)0}/d\nu|_{\nu_0}$ imply that $\delta_{(0)}{}^0$ rises above 45° somewhere below $E_{\mathrm{c.m.}} = 600$ MeV, and that $\delta_{(0)}{}^0$ rises to at least 60° below 700 MeV.[11] Solution A of Ref. 8 is typical of those solutions wherein $\delta_{(0)}{}^0$ does not quite reach 90° below 1000 MeV. Work subsequent to the publication of Ref. 8 has revealed the existence of solutions wherein $\delta_{(0)}{}^0$ is constrained to rise through 90° at a given mass m_σ, where m_σ can vary at least over the range from 700 to 1000 MeV. Fortunately, the contributions to λ and λ_1 are not very sensitive to the differences between these solutions for $\delta_{(0)}{}^0$. If $\delta_{(0)}{}^0$ remains below 90°, or if 700 MeV $\leq m_\sigma$, and if we neglect the S-wave contributions to λ and λ_1 coming from $E_{\mathrm{c.m.}}$ above 1250 MeV, then all of our solutions which incorporate Weinberg's predictions are such that

$$\lambda(S^0) \cong -1.4\lambda_1 m_\pi^{2} - 0.03 \pm 0.03,$$

$$\lambda(S^2) \cong -0.01 \pm 0.00,$$

$$\lambda_1(S^0) \cong 0.45\lambda_1 + (-0.11 \pm 0.003)m_\pi^{-2},$$

$$\lambda_1(S^2) \cong (0.001 \pm 0.001)m_\pi^{-2}.$$

Now let us add together all the aforementioned contributions to λ and λ_1, thus obtaining

$$\lambda_1 \cong (0.003 \pm 0.016)m_\pi^{-2} + 0.45\lambda_1 + 0.186\xi\Gamma(g)m_\pi^{-3}, \tag{4a}$$

$$\lambda \cong 0.04 \pm 0.05 - 1.4\lambda_1 m_\pi^{2} + 0.31\xi\Gamma(g)m_\pi^{-1}, \tag{4b}$$

where the indicated uncertainties are the roots of the sums of the squares of the contributing uncertainties. Since the uncertainties in Eqs. (4) stem primarily from the uncertainty in $\Gamma(\rho)$, they are correlated in sign.

Solving Eqs. (4) for λ and $\xi\Gamma(g)$ in terms of λ_1, we obtain

$$\lambda \cong -0.5\lambda_1 m_\pi^{2} + 0.04 \pm 0.03, \tag{5a}$$

$$\xi\Gamma(g) \cong 2.96\lambda_1 m_\pi^{3} - (0.016 \pm 0.086)m_\pi, \tag{5b}$$

which together with Weinberg's prediction (3a) for λ_1 imply that

$$\lambda \cong -0.01 \pm 0.03, \tag{6a}$$

$$\xi\Gamma(g) \cong (0.30 \pm 0.10)m_\pi. \tag{6b}$$

Obviously Eq. (6a) is consistent with Weinberg's prediction (3b) for λ; therefore, we proceed to compare the value (6b) for $\xi\Gamma(g)$ with experiment.

The reported values[12] for $\Gamma(g)$ vary over the range

(21 MeV or less) $\leq \Gamma(g) \leq 226$ MeV.

However, there is a tendency for the larger values of $\Gamma(g)$ to be accompanied by smaller values of ξ, so that there is wider agreement about the product $\xi\Gamma(g)$ than there is about $\Gamma(g)$. The g usually decays either into 2π or into 4π (including $g \to \rho\pi\pi \to 4\pi$ and $g \to 2\rho \to 4\pi$). Of the groups which have reported branching ratios, Johnston et al.[13] report $\Gamma(g) = (85 \pm 20)$ MeV, $(g \to 2\pi)/(g \to 4\pi) = 0.8 \pm 0.2$; Crennell et al.[14] report $\Gamma(g) \cong 100$ MeV, $(g \to 2\pi)/(g \to 4\pi) \geq 0.67$; and Biswas et al.[15] report $\Gamma(g) = (162_{-40}^{+58})$ MeV, $(g \to 2\pi)/(g \to 4\pi) < 0.4$. If we assume that the sum of the probabilities for $g \to 2\pi$ and $g \to 4\pi$ is 80%,[16] then the preceding experiments imply that $\xi\Gamma(g) = (0.22 \pm 0.09)m_\pi$, $\xi\Gamma(g) \geq 0.23m_\pi$, and $\xi\Gamma(g) < (0.27 \pm 0.09)m_\pi$, respectively. Thus the value (6b) is consistent with our present experimental knowledge of $\xi\Gamma(g)$, and we see that if the net contribution from spin states with $l \geq 4$ is negligible, then Weinberg's predictions (3) are consistent with the sum rules (1) and (2).

We remark that just as the g contributions to λ and λ_1 are substantial, so can the contributions from individual resonances with $l \geq 4$ be substantial.[17] However, the isospin-0 and -1 Regge recurrences do contribute to λ and λ_1 with opposite signs, so it is at least conceivable that the net contributions from $l \geq 4$ are indeed negligible. In addition, we remark that the value (6b) for $\xi\Gamma(g)$ is not crucial for consistency between Weinberg's predictions and the sum rules (1) and (2). Since the next higher resonance to be expected (f_0 recurrence) would contribute to λ and λ_1 with opposite sign from the g, as would a possible narrow isoscalar spin-2 resonance[18] at 1060 MeV, consistency might be possible if $\xi\Gamma(g)$ were somewhat larger than $(0.30 \pm 0.10)m_\pi$. If there were another $I=1$ Regge trajectory, then $\xi\Gamma(g)$ might be smaller.

While the $I=2$ sum rules (1) and (2) cannot be evaluated with sufficient accuracy at present to provide much information about λ and λ_1, we wish to emphasize that Weinberg's predictions receive considerable support from other sources Olsson[19] and quite recently Chu and Desai[20] have derived $I=1$ sum rules which support Weinberg's value for λ_1. Work by Tryon[8] and especially work by Olsson and Turner[21] has shown that Weinberg's values for λ and λ_1 are consistent with present data for the reaction $\pi N \to \pi\pi N$. In view of the aforementioned evidence, it seems likely that the high-energy contributions to the $I=2$ sum rules are indeed such as to render them consistent with Weinberg's predictions.

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[1]S. Y. Chu and B. R. Desai, Phys. Rev. Letters 21, 54 (1968).

[2]We have differentiated Eq. (3) of Ref. 1, and we use a slightly different notation.

[3]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[4]We neglect the $f_0'(1514)$ because it decays into 2π less than 14 % of the time.

[5]L. A. P. Balazs, Phys. Rev. 129, 872 (1963).

[6]Since the Breit-Wigner formula for $\mathrm{Im}A^{(l)I}$ does not vanish as $\nu \to \infty$ when $l \geq 1$, the integral for any single resonance contribution to λ for $l \geq 1$ diverges logarithmically. We have cut off the integrals roughly two full widths above each resonance mass; increasing the ρ and f_0 cutoffs by 200 MeV would change $\lambda(\rho)$ by +0.02, $\lambda(f_0)$ by -0.03, $\lambda_1(\rho)$ by $+0.005 m_\pi^{-2}$, and $\lambda_1(f_0)$ by $-0.009 m_\pi^{-2}$.

[7]The integrand for λ contains a factor $\sim d\nu'/\nu' \sim dE/E$, which favors the ρ over g by a factor ≈ 2. However, the factors $2l+1$ favor the g over ρ by 7/3. Both $P_l(\cos\theta)$'s ≈ 1 in the regions of interest. The integrand for λ_1 contains a factor $\sim d\nu'/\nu'^2 \sim dE/E^3$, which favors the ρ over g by a factor ≈ 10. However, the $(2l+1)$'s favor g over ρ by 7/3, and $dP_3(\cos\theta')/d\cos\theta' \approx 6 \times dP_1(\cos\theta')/d\cos\theta'$ in the regions of interest.

[8]E. P. Tryon, Phys. Rev. Letters 20, 769 (1968).

[9]Our general method of solution is presented in Ref. 8. For the present paper, we use 40-parameter trial functions instead of the six-parameter trial functions of Ref. 8, and we obtain solutions up to $E_{\mathrm{c.m.}} = 1250$ MeV instead of 900 MeV. The semiresonant solutions we obtain in this way are nearly identical below 900 MeV to those of Ref. 8. All solutions used for the present paper will be published elsewhere.

[10]The mutual consistency of Eqs. (3a)-(3c) depends on the amplitudes $A^I(\nu, \cos\theta)$ being linear functions of ν between ν_0 and 0. This linearity is well satisfied by our solutions. We assume $L \equiv (2a_0 - 5a_2)/6 = (0.105 \pm 0.015)m_\pi^{-1}$ and $a_0 = -\frac{7}{2}a_2 = (7/4)L$, and we find that $\lambda_1 \cong L m_\pi^{-1}$ within 5 %, and $\lambda \cong -(L/12)m_\pi + 0.005 \cong -0.004$.

[11]If we assume Eqs. (3) to be valid and use units wherein $m_\pi = 1$, then the dispersion relation for $\mathrm{Re}A^{(0)0}(\nu)$ is

$$\mathrm{Re}A^{(0)}(\nu) = 0.18 + 0.21\nu + \frac{\nu(\nu-\nu_0)}{\pi} \times P\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}A^{(0)0}(\nu')}{\nu'(\nu'-\nu_0)(\nu'-\nu)}$$

where $\mathrm{Im}A^{(0)0}(\nu')$ vanishes on the interval $-1 \leq \nu' \leq 0$. The integral is dominated by the ρ contribution to the left cut and by the absorptive part on the right cut. The ρ contribution to the integral is roughly given near threshold by $-0.50/(11+\nu)$. Since a_0 is small, the coefficient $\nu(\nu-\nu_0)/\pi$ is able to render small the contribution to $\mathrm{Re}A^{(0)0}(\nu)$ from the integral near threshold, so that $\mathrm{Re}A^{(0)0}(\nu) \approx (0.18 + 0.21\nu)$ near threshold. In this approximation, $\mathrm{Re}A^{(0)0}(\nu)$ reaches the unitarity limit of $\frac{1}{2}(1+1/\nu)^{1/2}$ at $\nu = 2.0$ (which corresponds to $\delta_{(0)}{}^0 = 45°$ at $E_{\mathrm{c.m.}} = 480$ MeV), and unitarity is violated for $\nu > 2.0$. When the contributions from the integral are included in $\mathrm{Re}A^{(0)0}(\nu)$, unitarity is preserved, but the unitarity limit is still reached below $\nu = 3.7$. Thus the solutions for $\mathrm{Re}A^{(0)0}(\nu)$ generated by our method are such that $\delta_{(0)}{}^0$ always rises through 45° somewhere below 600 MeV ($\nu = 3.7$); furthermore, $\delta_{(0)}{}^0$ continues rising to at least 60° before $E_{\mathrm{c.m.}}$ reaches 700 MeV.

[12]A. H. Rosenfeld et al., Rev. Mod. Phys. 40, 77 (1968); values and references are enumerated on p. 107 under the heading "ρ(1650)."

[13]T. F. Johnston et al., Phys. Rev. Letters 20, 1414 (1968).

[14]D. J. Crennell et al., Phys. Rev. Letters 18, 323 (1967).

[15]N. N. Biswas et al., Phys. Rev. Letters 21, 50

(1968). This group differentiates between the 2ρ, $\rho\pi\pi$, and 4π decay modes, reporting $(g\to 2\pi)/(g\to 2\rho) < 0.48$, and $[(g\to\rho\pi\pi)+(g\to 4\pi)]/(g\to 2\rho) < 0.37$. We have roughly combined these figures to obtain, in our terms, $(g\to 2\pi)/(g\to 4\pi) < 0.4$.

[16]Johnston et al., Ref. 13, report $(g\to\pi\omega)/(g\to 4\pi) = 0.25\pm0.10$. Crennell et al., Ref. 14, report $(g\to K\bar{K})/(g\to 2\pi) \leq 0.10$. Phase space suppresses the modes with six or more pions, so 80% seems a reasonable estimate.

[17]Since $dP_l(\cos\theta')/d\cos\theta' \approx l(l+1)/2$ for $\cos\theta' \approx 1$, remarks similar to those in Ref. 7 are applicable to any high-spin resonance; thus we conclude that the contributions to λ and λ_1 from any individual high-spin resonance can be substantial.

[18]D. H. Miller, L. J. Gutay, P. B. Johnson, V. Kenney, and Z. G. T. Guiragossián, Phys. Rev. Letters 21, 1489 (1968). However, other interpretations of the data seem possible; see K. W. Lai, Brookhaven National Laboratory Report No. 12611 (unpublished).

[19]M. G. Olsson, Phys. Rev. 162, 1338 (1967).

[20]S. Y. Chu and B. R. Desai, University of California Radiation Laboratory Report No. 18453, 1968 (unpublished).

[21]M. G. Olsson and Leaf Turner, Phys. Rev. Letters 20, 1127 (1968).

TEST OF AN ANSATZ FOR THE RESIDUE OF THE P' REGGE TRAJECTORY*

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The residue of the P' trajectory has been obtained from experimental data and found to be in agreement with the Ansatz of Barger and Phillips.

In a recent publication Barger and Phillips[1] interpreted the recurring minima in the t dependence of high-energy elastic-scattering cross sections by postulating a novel structure for the P' and ω Regge residues. In particular, for the P' amplitude,

$$f(P') = -\beta\nu^{\alpha}[1+\exp(-i\pi\alpha)]/\sin\pi\alpha, \tag{1}$$

they made the Ansatz that

$$\beta(t) \approx \lambda(t)\sin^2(\tfrac{1}{2}\pi\alpha) \tag{2}$$

instead of the usual Regge expression

$$\beta(t) = \gamma(t)G(\alpha)/\Gamma(\alpha+1). \tag{3}$$

Here $G(\alpha)$ represents the function of α required by the various ghost-suppressing mechanisms.[2] Insertion of Eq. (2) into Eq. (1) gives double zeros at $\alpha=0, -2, \cdots$, and no zeros at $\alpha=-1, -3, \cdots$, while the use of Eq. (3) yields zeros at $\alpha=-1, -3, \cdots$, and also possibly at even integral α, depending upon $G(\alpha)$. The form of Eq. (2) was suggested by assuming a linearly falling trajectory for the P' and translating the $d\sigma/dt$-vs-t data to $d\sigma/dt$ vs α.[1]

In an independent piece of work,[3] the form of the P' trajectory $\alpha(t)$ was deduced from experimental data and found to be consistent with a straight line out to $t=-4$ (GeV/c)^2. Also, the P' residue, although obtained in a somewhat more general model than that of Ref. 1, was found to have a t dependence qualitatively similar to Eq. (2). In this paper we make a detailed comparison of the model of Barger and Phillips with experimental data, and show that Eq. (2) is qualitatively correct and that Eq. (3) is inconsistent with experiment.

In addition, we obtain $\alpha(t)$ for the P' (in the approximation that the ρ contribution to elastic scattering can be neglected), and the t dependence of a fixed Pomeranchuk pole. The importance of the linearly falling trajectory has been emphasized by Chew,[4] and of the form of the P' residue by Hoff.[5]

As was done in Refs. 1 and 3 we assume a fixed Pomeranchuk pole with $f(P)=ai\nu$. We then have, for the differential cross section of π^+p or π^-p elastic scattering,

$$\begin{aligned} d\sigma/dt &= p_L^{-2}|ai\nu - \beta\nu^{\alpha}[1+\exp(-i\pi\alpha)]/\sin\pi\alpha|^2 \\ &= p_L^{-2}\{a^2\nu^2 + 2a\beta\nu^{\alpha+1} \\ &\quad + \beta^2\nu^{2\alpha}/\sin^2(\tfrac{1}{2}\pi\alpha)\}, \end{aligned} \tag{4}$$

where $\nu=(s-u)/4m$ in units of $\nu_0 = 1$ GeV and m is the nucleon mass. We wish to emphasize here that this is an extremely simplified expression. It assumes that the π^+p and π^-p cross sections are equal, i.e., that contributions of $I=1$ exchange (principally the ρ) can be neglected. This is qualitatively consistent with experiment except