ALGEBRAIC STRUCTURE OF SUPERCONVERGENCE RELATIONS*

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(Received 24 March 1969)

How to express superconvergence relations for pion scattering as statements about the generators X_a of chiral SU(2)⊗SU(2) is shown. One of the superconvergence relations has a remarkable algebraic significance: The matrix mJ_y (where m and $\vec{J}$ are the mass and spin matrices) is the sum of a chiral scalar and the fourth component of a chiral four-vector.

It has long been hoped that the various current-algebra and superconvergence sum rules would, when saturated with single-particle states, unite all hadrons in a general symmetry scheme.[1] Nevertheless, the only hadronic sum rules which have so far been expressed explicitly in a Lie-algebraic form are the chiral sum rules for massless-pion forward scattering; when saturated with single-particle states, these read[2]

$$[X_a,X_b]=i\epsilon_{abc}T_c, \tag{1}$$

$$X_a,[X_b,m^2]]\propto\delta_{ab}. \tag{2}$$

[Here T_c is the isospin matrix; a, b, and c run over 1, 2, and 3; m is the mass matrix $(m)_{\beta\alpha}\equiv m_\alpha\delta_{\beta\alpha}$; and $(X_a)_{\lambda'\beta,\lambda\alpha}$ is the reduced amplitude for the collinear process $\alpha\to\beta+\pi$ with $\vec{p}_\alpha$ and $\vec{p}_\beta$ in the z direction, defined in terms of the Feynman amplitude M by

$$M(\alpha\to\beta+\pi)\equiv 2F_\pi^{-1}(m_\alpha^2-m_\beta^2)\times(X_a)_{\lambda'\beta,\lambda\alpha}, \tag{3}$$

where $F_\pi\simeq 190$ MeV, λ and λ' are the helicities of α and β, and a is the pion isovector index.] Also, X_a is diagonal in helicity:

$$(X_a)_{\lambda'\beta,\lambda\alpha}=\delta_{\lambda'\lambda}(X_a(\lambda))_{\beta\alpha}. \tag{4}$$

These results tell us that the hadrons must for each helicity be assembled into representations of SU(2)⊗SU(2), but they offer no clue to how these representations are related for different helicities.

The purpose of this note is to emphasize that all of the superconvergence relations for pion scattering may also be expressed as statements about the matrix X_a. The absorptive part of any pion scattering amplitude is determined by the amplitudes for the noncollinear pion transitions $\alpha\to\beta+\pi$, and these amplitudes can be expressed in terms of X_a by boosting β to rest and then rotating so that α moves in the z direction. In particular, one of the superconvergence relations may be expressed as a new Lie-algebraic statement about X_a, which correlates different helicities:

$$[X_a,[X_b,mJ_y]]=\tfrac{1}{3}\delta_{ab}[X_c,[X_c,mJ_y]]. \tag{5}$$

Here $\vec{J}$ is the usual angular-momentum matrix, defined to act on helicity indices only. It follows[3] that the matrix mJ_y is the sum of a chiral scalar and the fourth component of a chiral four-vector. [Also, Eq. (4) says that X_a commutes with J_z; so Eq. (5) holds as well for the matrix mJ_x.]

One technical device which greatly aids in the derivation of such results is the use of lab-frame direct-channel helicity amplitudes rather than the usual center-of-mass-frame crossed-channel helicity amplitudes. Consider the scattering process $\alpha+\pi_a\to\beta+\pi_b$, with α at rest and β recoiling with fixed momentum in the z direction. The Feynman amplitude for this process may be regarded as a function $M_{\lambda'\beta b,\lambda\alpha a}(q)$ of the incoming pion four-momentum and is manifestly free of kinematic singularities in the components of $q\mu$. Also, invariance with respect to rotations about the z axis tells us that M equals a factor $(q_x\pm iq_y)^{|\lambda'-\lambda|}$ times a kinematic-singularity-free function of q_0, q_z, and $q_x^2+q_y^2$, all of which are polynomials in s. If we now choose our coordinate axes so that the pions move in the x-z plane, the transverse pion momentum q_x equals $[-st^2-t(s-m_\alpha^2)(s-m_\beta^2)]^{1/2}$ times a function of t, and the Feynman amplitude M takes the form

$$M_{\lambda'\beta b,\lambda\alpha a}=[-st^2-t(s-m_\alpha^2)(s-m_\beta^2)]^{\frac{1}{2}|\lambda-\lambda'|}F_{\lambda'\beta b,\lambda\alpha a}(s,t), \tag{6}$$

with F free of kinematic singularities in s for fixed t. By saturating the dispersion relation for F with single-particle states, boosting and rotating as indicated above, and using Eq. (3), we find (except for possible subtractions) that

$$F_{\lambda'\beta b,\lambda\alpha a}(s,t)=4F_\pi^{-2}\sum_{\gamma\sigma'\sigma}(s-m_\gamma^2)^{-1}(m_\beta^2-m_\gamma^2)(m_\alpha^2-m_\gamma^2)$$
$$\times[(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)+m_\gamma^2t]^{-\frac{1}{2}|\lambda-\lambda'|}d_{\lambda'\sigma'}^{(\beta)}[\chi_{\alpha\gamma}^{\beta}(t)-\pi]$$
$$\times[X_b(\sigma')]_{\beta\gamma}d_{\sigma'\sigma}^{(\gamma)}[-\theta_{\alpha\beta}^{\gamma}(t)][X_a(\sigma)]_{\gamma\alpha}d_{\sigma\lambda}^{(\alpha)}[\chi_{\beta\gamma}^{\alpha}(t)]\pm\text{crossed term}. \tag{7}$$

Here $d(\theta)$ is as usual $\exp(-iJ_y\theta)$, the crossed term is obtained by replacing s with u and interchanging a and b, and its sign is plus or minus according as $|\lambda-\lambda'|$ is even or odd. Also, χ and θ are given by[4]

$$\cos[\chi_{\beta\gamma}^{\alpha}(t)]=\frac{(m_\alpha^2-m_\beta^2)(m_\gamma^2-m_\alpha^2)+t(m_\gamma^2+m_\alpha^2)}{(m_\gamma^2-m_\alpha^2)[(m_\alpha^2-m_\beta^2)^2-2t(m_\alpha^2+m_\beta^2)+t^2]^{1/2}},$$

$$\sin[\chi_{\alpha\beta}^{\alpha}(t)]=\frac{2m_\alpha\sqrt{-t}}{m_\gamma^2-m_\alpha^2}\left[\frac{(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)+m_\gamma^2t}{(m_\alpha^2-m_\beta^2)^2-2t(m_\alpha^2+m_\beta^2)+t^2}\right]^{1/2},$$

$$\cos[\theta_{\alpha\beta}^{\gamma}(t)]=1+\frac{2m_\gamma^2t}{(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)},$$

$$\sin[\theta_{\alpha\beta}^{\gamma}(t)]=\frac{2m_\gamma\sqrt{-t}}{(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)}[m_\gamma^2t+(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)]^{1/2}.$$

We will restrict our attention here to the superconvergence relation for the amplitude:

$$[(s-m_\alpha^2)(s-m_\beta^2)+st]^{\frac{1}{2}(|\lambda-\lambda'|-1)}[F_{\lambda'\beta b,\lambda\alpha a}(s,t)]_{T=2}$$

with $|\lambda-\lambda'|$ odd. (Any matrix F_{ba} carrying a label $T=2$ is to be understood as $F_{ba}+F_{ab}-\frac{2}{3}\delta_{ab}F_{cc}$.) If the leading $T=2$, t-channel trajectory has $\alpha(t)<0$, then this amplitude vanishes faster than $1/s$ as $s\to\infty$ and therefore satisfies the superconvergence relation

$$0=\{\sum_{\gamma\sigma\sigma'}(m_\beta^2-m_\gamma^2)(m_\alpha^2-m_\gamma^2)[(m_\gamma^2-m_\alpha^2)(m_\gamma^2-m_\beta^2)+m_\gamma^2t]^{-\frac{1}{2}}d_{\lambda'\sigma'}^{(\beta)}[\chi_{\alpha\gamma}^{\beta}(t)-\pi]$$
$$\times[X_b(\sigma')]_{\beta\gamma}d_{\sigma'\sigma}^{(\gamma)}[-\theta_{\alpha\beta}^{\gamma}(t)][X_a(\sigma)]_{\gamma\alpha}d_{\sigma\lambda}^{(\alpha)}[\chi_{\beta\gamma}^{\alpha}(t)]\}_{T=2}. \tag{8}$$

Now let $t\to0$. For $m_\alpha>m_\beta$ the right-hand side of Eq. (8) is of order $\sqrt{-t}$ as $t\to-0$, and we find

$$0=\{[[X_a,[X_b,m^2]]_{T=2},mJ_y]-(m_\alpha^2-m_\beta^2)[X_a,[X_b,mJ_y]]_{T=2}\}_{\beta\lambda',\alpha\lambda}. \tag{9}$$

For $m_\alpha=m_\beta$ the right-hand side of Eq. (10) has a leading term of order unity as $t\to-0$, but this term vanishes by parity conservation; the next term is of order $\sqrt{-t}$ and gives[5]

$$0=\sum_{\sigma'\sigma}d_{\lambda'\sigma'}^{(\beta)}(-\tfrac{1}{2}\pi)\{[[X_a,[X_b,m^2]]_{T=2},J_y]+4m_\alpha[X_a,[X_b,mJ_y]]_{T=2}\}_{\beta\sigma',\alpha\sigma}d_{\sigma\lambda}^{(\alpha)}(\tfrac{1}{2}\pi). \tag{10}$$

In both Eqs. (9) and (10) the matrix elements would vanish automatically for $|\lambda-\lambda'|$ even; so these results hold for all λ and λ'. Also, the d functions in Eq. (10) are nonsingular matrices and can therefore be removed by multiplying with their inverses. Finally, Eq. (12) tells us that the first terms in Eqs. (9) and (10) vanish. Thus both (9) and (10) reduce to the desired result (5).

We will note just one example of the power of this result. Gilman and Harari[5] have proposed an attractive saturation scheme which includes the π, ρ, σ, A_1, D, δ, and χ^0 mesons. By taking matrix elements of Eq. (5) between A_1 states we find that the π-A_1 mixing angle is 90°, which would forbid the decay $\rho\to 2\pi$! Thus the algebraic constraints imposed by superconvergence relations cannot be met here without including more particles in this scheme.

There are an infinite number of other superconvergent relations, but I have not yet succeeded in expressing any of them as Lie-algebraic statements about the matrix X_a. Perhaps there is something special about the particular superconvergence relation which led to Eq. (5).

I am grateful for conversations with S. Fubini and F. Low.

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*Work supported in part through funds provided by the U. S. Atomic Energy Commission Contract No. AT(30-1)-2098.

[1]Much of the effort that has been expended on this program has been based on the use of off-mass-shell sum rules derived from the local algebra of currents, such as those of S. Fubini, Nuovo Cimento 43A, 475 (1966); and R. F. Dashen and M. Gell-Mann, in Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966, edited by A. Perlmutter, J. Wojtaszek, E. C. G. Sudarshan, and B. Kursunoǧlu (W. H. Freeman and Company, San Francisco, Calif., 1966). In my opinion these sum rules are of great importance in understanding the weak and electromagnetic interactions, but information about the hadron spectrum will have to come from purely hadronic–mass-shell sum rules derived from soft-pion theorems and Regge asymptotic behavior. New hope for progress with this more modest program has come from the remarkable quantization conditions imposed by soft-pion theorems in the Veneziano model; see C. Lovelace, Phys. Letters 28B, 265 (1968); M. Ademollo, G. Veneziano, and S. Weinberg, Phys. Rev. Letters 22, 83 (1969); S. Weinberg, to be published.

[2]S. Weinberg, Phys. Rev. (to be published). Equation (1) was first derived by the infinite-momentum method by Dashen and Gell-Mann, Ref. 1.

[3]The derivation is the same as given for the matirx m^2 by Weinberg, Ref. 2, Sec. IV.

[4]Despite appearances, the contribution of a state γ to F remains finite in the limit $m_\gamma\to m_\alpha$ or $m_\gamma\to m_\beta$.

[5]This is the saturated form of the "type-II superconvergence relation" derived by F. Gilman and H. Harari, Phys. Rev. 165, 1803 (1968). Its algebraic significance remained obscure because it was derived only for $m_\alpha = m_\beta$ and because its saturated form was expressed in terms of covariant coupling constants instead of the matrix X_a.

CLOSED EXPRESSIONS FOR THE ENERGY LEVELS IN BROKEN SYMMETRIES AND THE GENERAL FORM FOR REGGE DAUGHTER TRAJECTORIES*

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We propose a new method which summarizes in closed compact form all the implications of analyticity in a problem with broken symmetry. This method is applied to the Bethe-Salpeter equation to obtain the broken-SL(2, C) results for the Regge daughter trajectories in a closed functional form:

$$\alpha_k^{(\pm)}+k=F\left(t,t(\alpha_k^{(\pm)}+\tfrac{1}{2})^2,\pm E^{2M}(\alpha_k^{(\pm)}+M)(\alpha_k^{(\pm)}+M-1)\cdots(\alpha_k^{(\pm)}-M+1)\right),$$

where F is an analytic function in three variables.

Traditionally, broken symmetries have always been treated by a perturbation expansion in the symmetry-breaking parameter. Thus analyticity in this parameter has to be postulated together with other dynamical assumptions. Our aim is to explore all the implications of analyticity independent of dynamics in the most general form. We propose a method which summarizes all these informations in a closed compact form. This method can be applied to any representation of a symmetry group.

To illustrate the method, let us consider a simple quantum-mechanical example. If the rotational symmetry of a Hamiltonian H is broken by an external vector field $\vec{V}$, and if H is analytic in $\vec{V}$, then it has the following expansion:

$$H(\vec{V},\vec{x})=\sum_l(2l+1)H_l(V,r)V^lP_l(\cos\theta), \tag{1}$$