LOW ENERGY THEOREMS TO FOURTH ORDER IN e FOR COMPTON SCATTERING

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We prove low-energy theorems for Compton scattering, which are exact when the electromagnetic interaction is included up to fourth order and the strong interactions to all orders. These theorems give the terms of order $\omega^2 \ln\omega$, where ω is the photon c.m. momentum, besides establishing the classical low-energy theorem to the order e^4.

(1) Introduction. – The classical low-energy theorems[1] for Compton scattering give the exact scattering amplitude up to terms of the first order in the photon frequency ω. These theorems treat electromagnetism up to second order, i.e., $O(e^2)$ and strong interactions to all orders. These theorems were recently extended to higher orders[2] in ω. We propose here to extend these results in yet another direction, i.e., the inclusion of electromagnetism up to the fourth order, i.e., $O(e^4)$. If certain limits like Low's "infrared limit"[1] exist up to a certain order higher than e^2, then these classical theorems would be exact for the real part of the scattering amplitude up to that order. In fact our work shows that such is the case at least up to $O(e^4)$. Besides this result, which extends the validity of the known theorems to $O(e^4)$, we also obtain new theorems which give terms of the order $\omega^2 \ln\omega$.

Let $\langle c, d \,|\, T(s; \cos\theta) \,|\, a, b\rangle$ be the Compton scattering amplitude which describes, in the c.m. system, the process

target (a) + photon (b) → target (c) + photon (d),

where s is the c.m. energy and θ the c.m. scattering angle. Here a, b, c, and d refer to the respective helicities. Further, let

$$\langle c, d \,|\, T(s;\cos\theta) \,|\, a, b\rangle = [\cos\tfrac{1}{2}\theta]^{|\lambda+\mu|}[\sin\tfrac{1}{2}\theta]^{|\lambda-\mu|}\langle c, d \,|\, \hat{T}(s;\cos\theta) \,|\, a, b\rangle,$$

where $\lambda = a-b$, $\mu = c-d$, and let $\nu = (s - m_t^2)/2m_t$, m_t = target mass. We then have the following theorems for the pion Compton scattering:

$$\mathrm{Re}\langle 0, 1 \,|\, \hat{T}(s;\cos\theta) \,|\, 0, 1\rangle = -2e^2 - 2e^2(1-\cos\theta)\left(\frac{\nu}{m_t}\right) + \frac{e^4}{3\pi^2 m_t^2}(5-\cos\theta)\nu^2 \ln\nu + O(\nu^2), \tag{1}$$

$$\mathrm{Re}\langle 0, -1 \,|\, \hat{T}(s;\cos\theta) \,|\, 0, 1\rangle = -2e^2 + 2e^2(1+\cos\theta)\left(\frac{\nu}{m_t}\right) + \frac{e^4}{3\pi^2 m_t^2}(1-\cos\theta)\nu^2 \ln\nu + O(\nu^2), \quad 0<\theta<\pi, \tag{2}$$

and

$$\mathrm{Im}\langle 0, \pm 1 \,|\, \hat{T}(s;\cos\theta) \,|\, 0, 1\rangle = \left(\frac{e^4}{3\pi m_t}\right)\nu\left(1 - \frac{2\nu}{m_t}\right) + O(\nu^3). \tag{3}$$

It is interesting also to note that for the forward-scattering case, i.e., $\theta = 0$,

$$\mathrm{Re}\langle 0, 1 \,|\, \hat{T}(s;1) \,|\, 0, 1\rangle = -2e^2 + \left(\frac{4e^4}{3\pi^2 m_t^2}\right)\nu^2 \ln\nu + O(\nu^2) \tag{4}$$

[m_t = pion mass in Eqs. (1)-(4)]. One can obtain similar theorems for nucleon Compton scattering. We only mention here the results for the forward direction:

$$\tfrac{1}{2}\mathrm{Re}[\langle \tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, \tfrac{1}{2}, 1\rangle + \langle -\tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, -\tfrac{1}{2}, 1\rangle] = \frac{1}{2M}\left[-2e^2 + \frac{4e^4}{3\pi^2 M^2}\nu^2 \ln\nu + O(\nu^2)\right], \tag{5}$$

$$\tfrac{1}{2}\mathrm{Re}[\langle -\tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, -\tfrac{1}{2}, 1\rangle - \langle \tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, \tfrac{1}{2}, 1\rangle] = 2\mu^2\nu + O(\nu^3 \ln\nu), \tag{6}$$

$$\tfrac{1}{2}\mathrm{Im}[\langle -\tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, -\tfrac{1}{2}, 1\rangle + \langle \tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, \tfrac{1}{2}, 1\rangle] = \frac{e^4}{6\pi M^2}\nu\left(1 - \frac{2\nu}{M}\right) + O(\nu^3), \tag{7}$$

$$\tfrac{1}{2}\mathrm{Im}[\langle -\tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, -\tfrac{1}{2}, 1\rangle - \langle \tfrac{1}{2}, 1 \,|\, T(s;1) \,|\, \tfrac{1}{2}, 1\rangle] = \frac{e^4}{12\pi M^3}\nu^2\left(1 + \frac{6M\mu}{e}\right) + O(\nu^3), \tag{8}$$

where M is the nucleon mass and μ the anomalous nucleon magnetic moment. It is worthwhile emphasizing that the theorems for the forward spin-averaged amplitudes for both the $\pi\gamma$ and $N\gamma$ case have the same form, the extra factor $2M$ being due to the standard difference in the normalizations of the fermion and boson states. Fourth order in e spin-$\frac{1}{2}$ Compton scattering in the pure quantum electrodynamics (i.e., absence of strong interactions) was considered by Corinaldesi and Jost[3] for the spin-0 case and by Brown and Feynman[3] for the spin-$\frac{1}{2}$ case. Their expressions for $d\sigma/d\Omega$ are in agreement with our theorems.

In order to obtain the above results we work in an S-matrix framework. Recently, classical theorems have also been rederived in this framework.[4] We use unitarity, crossing, and only those fixed-$\cos\theta$ representations for which the convergence is assured by Froissart-Martin[5] type of bounds. In particular one does not have to assume a Regge-type asymptotic behavior.

(2) Pion Compton scattering. – This is described by two amplitudes $A(s;\cos\theta)$ and $B(s;\cos\theta)$ given by

$$\langle 0, 1 | \hat{T}(s;\cos\theta) | 0, 1\rangle = 2(s-m^2)^2 A(s;\cos\theta)$$

and

$$\langle 0, -1 | \hat{T}(s;\cos\theta) | 0, 1\rangle = [(s-m^2)^2/(2s)]B(s;\cos\theta)$$

(m = pion mass) which are kinematic-singularity free and thus can have only dynamical singularities. These singularities in s for fixed $\cos\theta$ can be easily worked out for the amplitudes A and B. They are, besides the contribution of a single-pion intermediate state, (i) a cut on the real axis for $s \geq m^2$ due to the direct channel, (ii) a cut on the real axis for $s \leq m^2$ due to the crossed $\pi\gamma$ channel, and (iii) a cut on the real axis for $s \leq 0$ due to the $\pi\pi \to \gamma\gamma$ channel intermediate states. The least massive state allowed here is a two-pion state. It is important to note that only the nonzero-mass intermediate states are allowed in the t channel as we are working only to $O(e^4)$, and hence the singularities arising from these exchanges are a finite distance away from $s = m^2$.

We further note that the major axis a of the Lehmann ellipse for both A and B is given by $a \to 1 + 2m^2/s$ for $s \to \infty$, and we can therefore, following Froissart-Martin type of arguments, establish the following asymptotic upper bounds:

$$|A(s;\cos\theta)| \underset{s\to\infty}{<} \text{const}(\ln^2 s)/s, \quad \text{for } \theta = 0, \pi;$$

$$|A(s;\cos\theta)| \underset{s\to\infty}{<} \text{const}(\ln^{3/2} s)/s^{5/4}, \quad \text{for } \pi > \theta > 0.$$

The similar bounds for $B(s;\cos\theta)$ are worse by a factor s.

We also note that both A and B are even under s,u crossing for t fixed, which leads to

$$A(s;\cos\theta) = A(s_c;\cos\theta_c), \tag{9}$$

where

$$s_c \equiv s_c(s,\theta) = 2m^2 - s + \frac{(s-m^2)^2}{2s}(1-\cos\theta), \tag{10}$$

$$\cos\theta_c \equiv \cos[\theta_c(s,\theta)] = \cos\theta + \frac{(s-m^2)[3s+m^2+(s-m^2)\cos\theta]\sin^2\theta}{[s+m^2+(s-m^2)\cos\theta]^2}, \tag{11}$$

and similarly for $B(s;\cos\theta)$.

The single-pion contributions to A and B, to be called A^π and B^π, respectively, are

$$A^\pi(s;\cos\theta) = \frac{B^\pi(s;\cos\theta)}{4m^2} = \frac{e^2}{(s-m^2)(u-m^2)}, \quad u \equiv 2m^2 - s + \frac{(s-m^2)^2}{2s}(1-\cos\theta). \tag{12}$$

Using the above information about the dynamic singularities, asymptotic bounds, and crossing, we can write the following representation for $A(s;\cos\theta)$:

$$[A(s;\cos\theta) - A^\pi(s;\cos\theta)] = \frac{1}{\pi}\int_{m^2}^\infty ds' \left\{\frac{\operatorname{Im}A(s';\cos\theta)}{s'-s} + \frac{\operatorname{Im}A(S_c(S',\theta);\cos\theta_c(s',\theta))}{s'+s-2m^2}\right\} \equiv \int_{m^2}^\infty I(s';s)ds'. \tag{13}$$

This representation is crucial for obtaining the various low-energy theorems. Using the classical low-energy theorems and noting that $\mathrm{Im}\,A(s';\cos\theta)$ for $s^h \geq s' \geq m^2$, where s^h is the lowest multiparticle hadronic state threshold, is given completely by the intermediate $\pi\gamma$ state, we can calculate $\mathrm{Im}A(s;\cos\theta)$ for s near m^2. Since the Compton scattering amplitudes up to $O(e^2)$ have a power series expansion in $(s-m^2)$ as $s\to m^2$, such a calculation would lead to a power series form for $\mathrm{Im}A(s;\cos\theta)$ also. The result of this calculation is given in Eq. (3) in terms of $\mathrm{Im}\langle 0,1|\hat{T}(s;\cos\theta)|0,1\rangle$, which is related to $\mathrm{Im}A(s;\cos\theta)$ simply by the factor $2(s-m^2)^2$. It can be easily verified that the calculated $\mathrm{Im}A(s;\cos\theta)$ is such that the representation (13) converges at the lower limit of the integration.

In order to calculate $\mathrm{Re}A(s;\cos\theta)$ by substituting the power series expansion of $\mathrm{Im}A(s;\cos\theta)$ in the above representation (13), we need the following results:

$$\int_{m^2}^{\infty} I(s';s)ds' = \int_{m^2}^{\lambda} I(s';s)ds' + O(1), \quad s\to m^2, \tag{i}$$

$$\int_{m^2}^{\lambda} \frac{(s'-m^2)^n}{s'-s}ds' = O(1) \text{ and } \int_{m^2}^{\lambda} \frac{(s'-m^2)^n ds'}{s'+s-2m^2} = O(1), \quad s\to m^2, \tag{ii}$$

where $m^2 < \lambda < s^h$ and $n \geq 1$.

Using these results we see that if we are interested in calculating the terms in $A(s;\cos\theta)$ which dominate $O(1)$ terms as $s\to m^2$, then it is sufficient to know $\mathrm{Im}A(s;\cos\theta)$ correct up to terms of $O(1)$ as $s\to m^2$. This we know from Eq. (3) obtained before. This then leads to Theorem (1). We thus see that the scattering amplitude up to the terms of order $\nu^2\ln\nu$ is well defined and does not suffer from the infrared divergence problem encountered in higher order electrodynamics. This is as it should be, since we know from an analysis of the Bloch-Nordsieck[6] type that the infrared-divergent term in the Compton amplitude has a coefficient $\nu^2(1-\cos\theta)$ as $\nu\to 0$.

(3) Forward nucleon Compton scattering.—Let

$$T_a(\nu) = \langle \tfrac{1}{2}, 1|T(s;1)|\tfrac{1}{2}, 1\rangle$$

and

$$T_p(\nu) = \langle -\tfrac{1}{2}, 1|T(s;1)|-\tfrac{1}{2}, 1\rangle$$

which have the crossing property

$$T_a(\nu) = T_p(-\nu) \tag{14}$$

and have the Froissart bounds

$$|T_{a,p}(\nu)| \underset{\nu\to\infty}{<} \mathrm{const}\,\nu\ln^2\nu.$$

Let us also note the single-nucleon intermediate-state contribution $T_{a,p}{}^N(\nu)$ to these amplitudes[4]:

$$T_p{}^N(\nu) = -e^2/M + 2\mu^2\nu, \tag{15}$$

$$T_a{}^N(\nu) = -e^2/M - 2\mu^2\nu. \tag{16}$$

Further, the forward amplitudes $T_p(\nu)/\nu^2$ and $T_a(\nu)$ are kinematic-singularity free.[7] Using the above information we can write the representation

$$T_p(\nu) - T_p{}^N(\nu) = \frac{\nu^2}{\pi}\int_0^\infty \frac{d\nu'}{\nu'^2}\left[\frac{\mathrm{Im}\,T_p(\nu')}{\nu'-\nu} + \frac{\mathrm{Im}\,T_a(\nu')}{\nu'+\nu}\right]. \tag{17}$$

This on again using crossing gives

$$T_a(\nu)-T_a^{N}(\nu)=\frac{\nu^2}{\pi}\int_0^\infty \frac{d\nu'}{\nu'^2}\left[\frac{\operatorname{Im}T_a(\nu')}{\nu'-\nu}+\frac{\operatorname{Im}T_p(\nu')}{\nu'+\nu}\right]. \tag{18}$$

It may be pointed out that we could not have written down this representation for $T_a(\nu)$ directly without using crossing and the representation (17) for $T_p(\nu)$ since we did not, a priori, know $[T_a(\nu)-T_a^{N}(\nu)]/\nu^2$ to be kinematic-singularity free.

We can again, as in the last section, using classical low-energy theorems obtain the theorems for $\operatorname{Im}T_a(\nu)$ and $\operatorname{Im}T_p(\nu)$ up to terms $O(\nu^2)$. These together with the above representations lead to the theorems on real parts of $T_a(\nu)$ and $T_p(\nu)$ given by Eqs. (5) and (6).

(4) Concluding remarks.—(i) If we wish to extend the present theorems to $O(e^6)$, we will have to contend with the 2γ intermediate state in the t channel among other complications. The presence of this state makes the Lehmann ellipse shrink completely to real axis. We would therefore no longer be able to establish the upper bounds at least by using the standard method.

(ii) It appears that for an arbitrary-spin target the following conjecture may be true:

Conjecture: Let the spin-averaged forward Compton amplitude for a target (charge e, mass m_t) be $[T(s)]_{\text{av}}$. Then

$$\operatorname{Re}[T(s)]_{\text{av}}=[T(m_t^2)]_{\text{av}}\left[1-\frac{2e^2}{3\pi^2 m_t^2}\nu^2\ln\nu+O(\nu^2)\right],$$

where $[T(m_t^2)]_{\text{av}}$ is given by the Thomson theorem.

(iii) The $O(e^4)$ low-energy theorems for the nucleon Compton scattering amplitude for the case of general scattering angle and other details will be discussed in a later communication.

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