of Ref. 2 used in Ref. 3. Neither Ref. 2 nor fit $(\text{Y-IV})_{pp+np}$ has made use of the Saclay measurements.

The difference in the quality of fit between the dashed curve and the full one may be partly accidental but the fact that the Livermore curve is systematically high appears unlikely to be explicable that way. The ratio of chi-square values corresponding to the two fits is 2.85/0.76 = 3.7. The normalization factor which in the Yale notation[6] is A_δ^j is 1.0036 and its effect on χ^2 is a factor $\approx 1-0.0013$. Systematic differences in the scale of the expected and observed $P(\theta)$ are thus practically absent in the case of $(\text{Y-IV})_{pp+np}$. The degree to which this is the case is doubtless accidental but the Saclay measurements indicate that the spin-orbit interaction is well represented by the last mentioned fit at 20 MeV in p-p scattering. The comparisons made above indicate the desirability of ascertaining the reason for the difference between the Berkeley[1] and the Saclay[3] experimental results.

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DIFFRACTED WAVE FIELDS EXPRESSIBLE BY PLANE-WAVE EXPANSIONS CONTAINING ONLY HOMOGENEOUS WAVES*

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A comprehensive study of the properties of "source-free" wave fields (i.e., diffracted wave fields expressible by plane-wave expansions containing only homogeneous waves) is summarized. The theorems proved in the study are stated. A new series mode expansion for "source-free" wave fields in included.

Although the wave fields herein called "source-free" wave fields[1] have played an important role in the theories of propagation and diffraction of monochromatic, scalar waves for nearly two decades,[2] their unique properties have not been studied in detail. Recent advances[3-12] in the diffraction theories of imaging and holography have indicated a need for improved understanding of these fields. "Source-free" wave fields are important because (1) most scalar wave fields that occur in diffraction theory can be approximated by "source-free" wave fields in regions of space far from sources, and (2) "source-free" wave fields can be treated by much simpler mathematical techniques than can other fields. In the vicinity of a source, however, it is important to distinguish between "source-free" wave fields and wave fields that are not "source-free" since the behaviors of these two types of wave fields differ greatly in the vicinity of a source. Hence, indiscriminate use of "source-free" wave fields in the study of images of sources can lead to erroneous results. The results of a recent study of the properties of "source-free" wave fields are summarized here; a comprehensive treatment, complete with rigorous proofs of the theorems, will follow.

Consider a wave field $u(x,y,z)$ expressed by the angular spectrum representation[13]

$$u(x,y,z)=\iint_{-\infty}^{\infty}F(p,q)H(p,q,x,y,z)dpdq \quad \text{for } z>0, \tag{1}$$

where

$$H(p,q,x,y,z)=G(p,q,z)\exp[2\pi i(px+qy)], \tag{2}$$

$$G(p,q,z)=\exp(2\pi i\,|\bar{k}-p^2-q^2|^{1/2}z) \ \text{ for } p^2+q^2\le\bar{k}^2, \tag{3a}$$

$$G(p,q,z)=\exp(-2\pi\,|p^2+q^2-\bar{k}^2|^{1/2}z) \ \text{ for } p^2+q^2>\bar{k}^2. \tag{3b}$$

$\bar{k}$ is equal to $k/2\pi$, k is the propagation constant of the field, and the angular spectrum $F(p,q)$ is a square-integrable function that characterizes the field. The validity of this representation has been discussed by Montgomery[7] and by Lalor.[14] It can be shown that if $F(p,q)\in L_2$ and if $u(x,y,z)$ is given by (1) for $z>0$, then for $z>0$, $u(x,y,z)$ is continuous with respect to each variable x,y,z and has partial derivatives of all orders continuous with respect to each variable. Moreover, $u(x,y,z)$ satisfies the Helmholtz equation

$$\nabla^2 u+k^2 u=0 \tag{4}$$

for $z>0$ and the boundary condition

$$\lim_{z\to 0} u(x,y,z)=f(x,y), \tag{5}$$

where $f(x,y)$ is the inverse Fourier transform of $F(p,q)$ given by

$$f(x,y)=\lim_{P\to\infty}\int_{-P}^{P}\int_{-P}^{P}F(p,q)\exp[2\pi i(px+qy)]dpdq \tag{6}$$

and lim means the limit in the L_2 norm.[15]

Equation (1) expresses $u(x,y,z)$ as a superposition of plane waves $H(p,q,x,y,z)$ with amplitudes $F(p,q)$. The branch of the square root given in (3) is the appropriate one to use when the sources of the field are confined to the region $z\le 0$. The plane waves are called homogeneous when $p^2+q^2\le\bar{k}^2$ and inhomogeneous or evanescent when $p^2+q^2>\bar{k}^2$. Since the inhomogeneous waves decay exponentially as z increases, they are frequently neglected in theoretical studies, and the integration in (1) is carried out only over the region $p^2+q^2\le\bar{k}^2$.

Let $u(x,y,z)$ be given by (1) for $z>0$ with $F(p,q)\in L_2$ and let $f(x,y)$ be given by (6). We say that $u(x,y,z)$ is "source free" if and only if $F(p,q)$ vanishes almost everwhere for $p^2+q^2>\bar{k}^2$. Then, the following results are ture.

Theorem I.—If $u(x,y,z)$ is source free, then

$$\lim_{z\to 0} u(x,y,z)=f(x,y). \tag{7}$$

Theorem II.—If $u(x,y,z)$ is source free, (1) can be used to extend $u(x,y,z)$ into the region $z\le 0$ to obtain a continuous, bounded solution of (4) for all space.

Theorem III.—For given $f(x,y)\in L_2$, $u(x,y,z)$ is source free if, and only if, $f(x,y)$ is equal almost everywhere to a function $f_0(x,y)$ that can be extended to the whole space of two complex variables $X=x+ix'$, $Y=y+iy'$ as an entire function $f_0(X,Y)$ such that

$$|f_0(X,Y)|\le A\exp[k(x'^2+y'^2)^{1/2}], \tag{8}$$

where x',y' are real variables and A is a positive constant.

Theorem IV.—Equation (1) can be used to extend $u(x,y,z)$ to the whole space of three complex variables $X=x+ix'$, $Y=y+iy'$, $Z=z+iz'$ as an entire function $u(X,Y,Z)$ such that for constant z'

$$|u(X,Y,Z)|\le B\exp[k(x'^2+y'^2)^{1/2}] \tag{9}$$

(where x',y' are real variables and B is a positive constant that can depend on z'), if, and only if, $u(x,y,z)$ is source free.

Theorem V.[16]—On planes of constant z, the two-dimensional autocorrelation function

$$\rho(\xi,\eta,z)=\iint_{-\infty}^{\infty}u(x,y,z)u^*(x+\xi,y+\eta,z)dxdy \tag{10}$$

of $u(x,y,z)$ is independent of $z>0$ and for all real ξ,η if, and only if, $u(x,y,z)$ is source free. (The asterisk indicates the complex conjugate operation.)

Theorem VI.[17]– Let $F(p,q)G(p,q,z)\in L_1$ for $z\geq -K$ where K is a positive constant, and let $u(x,y,z)$ be given by (1) for $z\geq -K$. Then the field $u'(x,y,z)$ given by (1) for the boundary condition

$$\lim_{z\to 0} u(x,y,z)=f^*(x,y) \tag{11}$$

is $u'(x,y,z)=u^*(x,y,-z)$ for all $z>0$ if, and only if, $u(x,y,z)$ is source free.

Theorem VII.–If $F(p,q)=0$ almost everywhere for $p^2+q^2>(k-\delta)^2$ and some $\delta>0$, then $u(x,y,z)$ can be expressed by the series mode expansion

$$u(x,y,z)=\sum_{m,n=0}^{\infty}\frac{a_{mn}}{(2\pi i)^{m+n}}\left[\frac{\partial^{m+n}}{\partial p^m\partial q^n}H(p,q,x,y,z)\right]_{p=q=0}, \tag{12}$$

where

$$a_{mn}=\frac{1}{m!n!}\left[\frac{\partial^{m+n}}{\partial x^m\partial y^n}f(x,y)\right]_{p=q=0}. \tag{13}$$

Discussion of the significance and interpretation of the results given in the theorems is deferred to the more comprehensive paper to follow.

Equation (12) can be used to represent wave fields that cannot be expressed by (1) or any of the usual diffraction formulas. The series in (12) is absolutely and uniformly convergent for all x,y,z if $f(x,y)\in B$ where B is the space of all functions $f(x,y)$ that can be extended to the whole space of two complex variables $X=x+ix'$, $Y=y+iy'$ as an entire function $f(X,Y)$ such that

$$|f(X,Y)|\leq A\exp[(k-\delta)(|X|^2+|Y|^2)^{1/2}] \tag{14}$$

for some $\delta>0$. If $f(x,y)\in B$, the series in (12) can be rewritten in the form of a Taylor series in x,y,z and in the form

$$u(x,y,z)=(\tfrac{1}{2}\pi kz)^{\frac{1}{2}}\sum_{n=0}^{\infty}\frac{H_{n-\frac{1}{2}}^{(1)}(kz)}{n!}\left(\frac{-z}{2k}\right)\Delta_2^{\,n}f(x,y), \tag{15}$$

where Δ_2^n is $(\partial^2/\partial x^2+\partial^2/\partial y^2)^n$ expanded with $\partial^2/\partial x^2$ and $\partial^2/\partial y^2$ treated as algebraic quantities and where $H_{n-\frac{1}{2}}^{(1)}(kz)$ is the Hankel function of the first kind. The series in (15) was derived earlier by Bremmer[18] and recently by Lalor[4]; different approaches were used. Our results show that (12) and (15) give a vaild solution to (4) and (7) for all $f(x,y)\in B$.

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[1]"Source-free" wave fields are defined later in the text. The terminology "source-free" was chosen because a source-free wave field can be extended as a wave field in all space with no sources anywhere (even at points infinitely far away). See Theorem II.

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CURRENT ALGEBRA AND THE PION TRAJECTORY

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It is shown that one conventional interpretation of the success of current-algebra results requires the pion Regge trajectory to choose the Toller quantum number $M=0$ at zero energy.

There has been speculation that the Toller [O(3,1)] classification of the pion Regge trajectory gives restrictions on the Regge residue functions which can be compared with certain predictions of current algebra.[1,2] The purpose of this note is to show that one conventional interpretation of current-algebra results requires the pion trajectory to choose the Toller quantum number $M=0$.[3] The converse is not true; the $M=0$ assignment enforces no constraint, current algebraic or otherwise, on amplitudes involving pions.

We consider first the possibility that the pion trajectory belongs to an $M=0$ or $M=1$ conspiracy class.[4] The $P=(-1)^{J+1}$ (pion) member of the conspiracy contributes to the t-channel reactions, $\bar{N}N\to\sigma\pi$,[5] or $\bar{N}N\to\rho\pi$ with zero-helicity ρ's, through an amplitude which we denote in either case by $f_{1/2,1/2,0,0}$. As t approaches zero in either reaction there is a conspiracy condition,[6]

$$f_{\frac{1}{2},\frac{1}{2},0,0}(s,t) -(i/\sin\theta_t)f_{\frac{1}{2},-\frac{1}{2},0,0}(s,t)\sim\sqrt{t}. \quad (1)$$

One can easily see that in either the $M=0$ or the $M=1$ case the term involving $f_{1/2,-1/2,0,0}$ has no s^α contribution.[7] Thus, the amplitude $f_{1/2,1/2,0,0}$ behaves like $t^{\frac{1}{2}}s^\alpha$ for large s, small t. This leads to a behavior of the residue function for small t of the form

$$\beta_{1/2,1/2,0,0}(t)=\beta_{1/2,1/2}^{\bar{N}N}\beta_{0,0}^{\rho\pi\text{ or }\sigma\pi} \sim t^{-\frac{1}{2}\alpha+\frac{1}{2}}. \quad (2)$$

At the equal-mass $\bar{N}N$ vertex the normal couplings are known from O(3, 1) consideration for the $P=(-1)^{J+1}$ part of the $M=0$ and $M=1$ trajectories:

$$M=1:\ \beta_{1/2,1/2}\sim\text{const};$$
$$M=0:\ \beta_{1/2,1/2}\sim\sqrt{t}. \quad (3)$$

Thus, we read off the maximal behavior of the residue function in the $\sigma\pi$ (or $\rho\pi$) system as

$$M=1:\ \beta_{0,0}(t)\sim t^{\frac{1}{2}-\frac{1}{2}\alpha};$$
$$M=0:\ \beta_{0,0}(t)\sim t^{-\frac{1}{2}\alpha}. \quad (4)$$

Next we imagine a limit in which the pion mass approaches zero. In the limit $\alpha(0)\to0$ and $t\to0$ the β functions in (4) become the actual amplitudes for $\pi\pi\to\sigma$ and $\pi\pi\to\rho$, in our world in which (at least) one incident π has zero mass. We see that for the $M=1$ case the amplitudes vanish as the pion mass; for the $M=0$ case they remain