in which the spatial dependence of the interaction is taken from the electromagnetic form factor of the proton.[7] In addition to the absorptive part of the interaction coming from the form factor, a real part, corresponding to refraction in an optical model, and a spin-orbit interaction are included in the model. The spatial dependence of the spin-orbit interaction is rather arbitrary and was chosen so as to fit the high-energy pp polarization data which are available for $-t<0.8$ $(\mathrm{GeV}/c)^2$.[2] The solid curve in Fig. 1 shows the prediction of the model. This curve differs considerably from curve a in Fig. 2 of Ref. 7. According to the authors, the real part of the scattering amplitude was not included in a consistent way in their calculation of the polarization. The solid curve in our Fig. 1 is the result of their corrected calculation.[8] The overall normalization of the polarization in the model is arbitrary. We chose the normalization to give $P=0.2$ at the peak at small $|t|$.

The model is, very qualitatively, in agreement with the data. Both exhibit a dip, with the polarization becoming somewhat larger at large momentum transfers than it is in the forward peak.

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RENORMALIZATION GROUPS

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General renormalization transformations are defined. In case they build up a group, the renormalization group, some of its properties are indicated and finally some physical consequences are sketched.

The renormalization group has been widely used for the determination of the asymptotic behavior of Feynman graphs or for the summation of certain classes of such graphs.[1] In this note, we want to build up a general scheme, to define in its full generality the renormalization group and sketch some of its physical applications. We consider the mth radiative correction of the time-ordered product of N field operators:

$$\Delta^{N,m}(x,\kappa)\equiv\Delta^{N,m}(x_1\cdots x_N,\kappa), \tag{1}$$

where the $\Delta^{N,m}$ are Lorentz scalars and κ is a parameter of the theory, the so-called squared mass ($\kappa=m^2$) of the particles of the field. We suppose furthermore that we are dealing with regularized $\Delta^{N,m}$ (by a spreading in space and time of the interaction) or with a renormalized theory; in any case, the $\Delta^{N,m}$ will be either analytic functions of the regularization parameter or tempered distributions in the variables x.

Starting from the set $\{\Delta^{N,m}(x,\kappa)\}$, we shall expand the vacuum expectation value $\langle 0|T\{\varphi(x_N)\cdots \times\varphi(x_1)\}|0\rangle$ as a formal series in another parameter g, the so-called coupling constant of the theory:

$$\langle 0|T\{\varphi(x_N)\cdots\varphi(x_1)\}|0\rangle \equiv \Delta^N(x,g,\kappa) = \sum \frac{g^m}{m!}\Delta^{N,m}(x,\kappa). \tag{2}$$

(1) Consider a set of transformations $\{\Lambda\}$ acting on the Δ^N's,

$$\Delta^N(x,g,\kappa) \to \Delta_\Lambda{}^N(x,g,\kappa) = \Lambda\Delta^N(x,g,\kappa)$$

such that they replace any given radiative correction $\Delta^{N,s}$ by a finite linear combination of other radiative corrections and their derivatives with respect to κ, i.e.,

$$\Delta^{N,s}(x,\kappa) \to \Delta_\Lambda{}^{N,s}(x,\kappa) = \sum_{0\le m,q<\infty} C_{m,s}{}^{N,q}(\Lambda)\partial^q\Delta^{N,m}(x,\kappa) \equiv \mathcal{R}(\Lambda)\Delta^{N,s}(x,\kappa), \tag{3}$$

where ∂ is the derivative with respect to the squared mass κ; the $C_{m,s}{}^{N,q}(\Lambda)$ are numerical coefficients, the <u>structure coefficients</u>; and the linear operator $\mathcal{R}(\Lambda)$ is defined by (3). As an example of a Λ transformation, we may take any of the transformations (12), (18), or (21) below. Under such a transformation, to be called a <u>general renormalization transformation</u>, the full vacuum expectation value as defined in (2) becomes

$$\Delta^N(x,g,\kappa) \to \Delta_\Lambda{}^N(x,g,\kappa) = \mathcal{R}(\Lambda)\Delta^N(x,g,\kappa) = \sum \frac{g^s}{s!}\mathcal{R}(\Lambda)\Delta^{N,s}(x,\kappa)$$

$$= \sum_{s,m,q} \frac{g^s}{s!} C_{m,s}{}^{N,q}(\Lambda)\partial^q\Delta^{N,m}(x,\kappa). \tag{4}$$

(2) Suppose now that among all possible renormalization transformations, we choose a set of transformations Λ which build up a group: a <u>renormalization group</u>. Such are, for instance, any one of the groups defined by (12), (18), and (21).

Let us look for necessary conditions under which the linear operators $\mathcal{R}(\Lambda)$ constitute a representation of this group. The existence of the product property $\mathcal{R}(\Lambda_2)\mathcal{R}(\Lambda_1) = \mathcal{R}(\Lambda_2\Lambda_1)$, where $\Lambda_2\Lambda_1$ is the product of the operations Λ_1 and Λ_2, and the existence of the inverse $\mathcal{R}(\Lambda)^{-1} = \mathcal{R}(\Lambda^{-1})$ lead to the equations

$$\sum_{m,0\le q'\le q} C_{s,m}{}^{N,q-q'}(\Lambda_2)C_{m,s'}{}^{N,q'}(\Lambda_1) = C_{s,s'}{}^{N,q}(\Lambda_2\Lambda_1), \tag{5}$$

$$\sum_{m,0\le q'\le q} C_{s,m}{}^{N,q-q'}(\Lambda)C_{m,s'}{}^{N,q'}(\Lambda^{-1}) = C_{s,s'}{}^{N,q}(I) = \delta_{q0}\delta_{ss'}, \tag{6}$$

where the structure coefficients are supposed to be independent of κ. We note furthermore that by means of these coefficients we may obtain some of the matrix representations of the renormalization group.

It may happen that the transformations $\{\Lambda_k\}$, which were the starting point of our considerations, do not constitute a group, but are such that, for any k, the inverse $\Lambda_k{}^{-1}$ does exist. It is then useful to consider the <u>semigroup of renormalization</u> built up by the transformations

$$L_{kk'} = \Lambda_k\Lambda_{k'}{}^{-1}; \tag{7a}$$

and one verifies that

$$L_{k_nk_{n-1}}L_{k_{n-1}k_{n-2}}\cdots L_{k_2k_1} = L_{k_nk_1}, \tag{7b}$$

which expresses precisely the semigroup property.

Turning back to the $\{\mathcal{R}(\Lambda_k)\}$ which in this case are supposed to satisfy (6) but not (5), we may build up a representation of this semigroup through the mapping

$$L_{kk'} \to R(\Lambda_k, \Lambda_{k'}) = \mathcal{R}(\Lambda_k)\mathcal{R}(\Lambda_{k'})^{-1}, \tag{8}$$

where R is a function of Λ_k and $\Lambda_{k'}$ (not of the product $\Lambda_k \Lambda_{k'}$) and (8) is a necessary condition for a representation of our semigroup.

(3) We want now to sketch some physical applications of the former remarks by defining some commutative renormalization groups.

(a) One of the renormalization groups, the Z group, may be defined as the commutative group of scale transformations of the propagators:

$$\Delta^N(x,g,\kappa) \to \Delta_Z{}^N(x,g,\kappa) = Z^{-\frac{1}{2}N}\Delta^N(x,g,\kappa), \tag{12}$$

where Z is a formal series in g with 1 as a first term. Defining the structure coefficients $C_{m,s}{}^{N,0}$ by

$$Z^{-\frac{1}{2}N}\frac{g^m}{m!} = \sum_{s \geq m} \frac{g^s}{s!} C_{m,s}{}^{N,0}(Z), \tag{13}$$

one verifies that

$$\Delta^{N,s}(x,\kappa) \to \Delta_Z{}^{N,s}(x,\kappa) = \sum_{m \leq s} C_{m,s}{}^{N,0}(Z)\Delta^{N,m}(x,\kappa). \tag{14}$$

Formula (14) shows that the linear combination of radiative corrections which is substituted for the $\Delta^{N,s}(x,\kappa)$ involves all mth radiative corrections with $m \leq s$, and furthermore, that one gets a representation of this group by matrices with elements $C_{m,s}{}^{N,0}(Z)$.

One may also realize such a transformation through the definition of a renormalization point, i.e., after Fourier transforming the Δ^N, one chooses an arbitrary four-vector λ such that

$$\Delta_\lambda{}^N(p,g,\kappa) = \frac{(p_1{}^2+\kappa)\cdots(p_N{}^2+\kappa)\Delta^N(p,g,\kappa)}{(\lambda^2+\kappa)\cdots(\lambda^2+\kappa)\Delta^N(\lambda,g,\kappa)}. \tag{15}$$

The renormalization group becomes then a semigroup where

$$\Delta_\lambda{}^N(p,g,\kappa) = R(\lambda_n, \lambda_m)\Delta_{\lambda_m}{}^N(p,g,\lambda), \tag{16}$$

with

$$R(\lambda_n, \lambda_m) = \mathcal{R}(\lambda_n)\mathcal{R}(\lambda_m)^{-1}, \tag{17}$$

the operator $\mathcal{R}(\lambda)$ being defined through structure coefficients whose expression is analogous to the one given by (15).[2]

(b) Another interesting group, the Z,z group, is the one corresponding to the transformation

$$\Delta^N(x,g,\kappa) \to \Delta_{Z,z}{}^N(x,g,\kappa) = Z^{-\frac{1}{2}N}\Delta^N(x,zg,\kappa), \tag{18}$$

where Z,z are formal series in g with 1 as a first term. The structure coefficients are defined by

$$Z^{-\frac{1}{2}N}\frac{(zg)^m}{m!} = \sum_{s \geq m} \frac{g^s}{s!} C_{m,s}{}^{N,0}(Z,z) \tag{19}$$

and

$$\Delta^{N,s}(x,\kappa) \to \Delta_{Z,z}{}^{N,s}(\kappa) = \sum_{m \leq s} C_{m,s}{}^{N,0}(Z,z)\Delta^{N,m}(x,\kappa). \tag{20}$$

We may again, as in (a), form matrix representations of the renormalization Z, z group, and for theories like quantum electrodynamics which admit a Ward identity, one may draw important conclusions for the asymptotic behavior of the propagators and for the summation of classes of Feynman graphs.

(c) We finally may consider the full renormalization group:

$$\Delta^N(x,g,\kappa) \to \Delta_{Z,z,\delta\kappa}{}^N(x,g,\kappa) = Z^{-\frac{1}{2}N}\Delta^N(x,zg,\kappa-\delta\kappa), \tag{21}$$

where $Z, z, \delta\kappa$ are formal series in g. We suppose the first terms of the Z and z series to be 1, while the first term of the series $\delta\kappa$ is of the form $g^n\delta\kappa_n$ ($n \geq 1$). We define as before the structure coefficients:

$$Z^{-\frac{1}{2}N}\frac{(zg)^m}{m!}\frac{(-\delta\kappa)^q}{q!} = \sum_{s \geq m+q}\frac{g^s}{s!}C_{m,s}{}^{N,q}(Z,z,\delta\kappa); \tag{22}$$

and denoting by ∂ the derivative with respect to κ, one gets the following transformation law:

$$\Delta^{N,s}(x,\kappa) \to \Delta_{Z,z,\delta\kappa}{}^{N,s}(x,\kappa) = \sum_{\substack{m,q\\ m+q \leq s}} C_{m,s}{}^{N,q}(Z,z,\delta\kappa)\partial^q\Delta^{N,m}(x,\kappa), \tag{23}$$

which corresponds to the general renormalization transformation. In both cases (b) and (c), as in (a), one may introduce the renormalization semigroup.

(d) We finally point out some important connections with renormalization theory.

First of all, one may build up a renormalization theory starting from a renormalization transformation (21) and showing that for certain classes of theories, there exists a possible choice of $Z, z, \delta\kappa$ (which are now functions of κ) such that, when regularization is removed, the $\Delta_{Z,z,\delta\kappa}{}^{N,s}(x,\kappa)$ become tempered distributions in x. The former choice then determines completely the structure coefficients.[3]

One may also remark that for any given Feynman diagram, Bogoliubov's rule for the definition of its finite part is precisely a method for the definition of the structure coefficients. Then, renormalizable field theories are the ones where a finite number of changes in Δ^N (changes in scale, changes of the parameters, etc.) are represented by a set of structure coefficients such that the renormalized $\Delta^{N,m}$'s given by (23) become tempered distributions.

We note also that, depending on the renormalization point (external momenta on the mass shell or taken to be 0), one may define classes of representations of renormalizable field theories; the study of their equivalence (or inequivalence) is of physical importance as is the determination of the invariants and covariants of these groups.

These points and others related to the physical meaning of the renormalization group will be fully studied in a forthcoming paper.

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