of matter and antimatter, such as the theory of Alfvén and Klein.[12]

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LORENTZ POLES IN EQUAL-MASS SCATTERING FROM ANALYTICITY AND FACTORIZATION*

J. B. Bronzan and C. Edward Jones

Laboratory for Nuclear Science and Physics Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

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It is shown, by means of factorization, that the daughter Regge trajectories implied by analyticity in unequal-mass scattering amplitudes constitute a single Lorentz pole when they couple to an equal-mass channel. At zero energy the equal-mass elastic amplitude has the form const $\times D_{-\alpha(0)-2}^{\,1}(-z)$.

In the past few years, two different approaches have been used to study the properties of Regge poles and S-matrix elements at zero total energy ($u=0$). The first involves analyticity[1] and the second uses group theory.[2] The analyticity approach can be employed when the particles in the u channel have unequal masses (like pion-nucleon scattering). In this case an apparent conflict arises between the presence of poles in the angular-momentum plane and the requirement of analyticity in the total energy at $u=0$. The conflict is resolved by introducing a set of daughter Regge trajectories at $\alpha(0)-k$ ($k=1,2,\cdots$), where $\alpha(0)$ is the leading or parent trajectory.

If the masses in the u channel are pair-wise equal, the foregoing conflict with analyticity does not arise, but one may invoke the O(4) [or O(3, 1)] symmetry of the scattering amplitude which results from the vanishing at $u=0$ of Q_u, the total center-of-mass four-momentum in the u channel. Poles in the u channel are classified according to irreducible representations of O(4), and then each such Lorentz pole $n(0)$ gives rise to a sequence of daughter Regge poles with the parent trajectory located at $\alpha(0)=n(0)$. The daughter sequence lies below $\alpha(0)$ with a spacing of two units, $\alpha(0)-2n$ ($n=1,2,\cdots$), if the scattering particles are spinless, and a spacing of one unit if the particles have spin. The ratios of the residues of the daughter trajectories are also determined by expanding the Gegenbauer function in terms of Legendre functions. On the other hand, the group theoretic approach cannot be immediately applied to the scattering amplitudes for unequal-mass particles, essentially because the center-of-mass three-momentum becomes infinite at $u=0$.

It is naturally desirable to attempt to bridge the gap between the analyticity and group theoretic treatments of daughter trajectories, and to see if the analyticity and group-theory daughters are really the same in some sense. Some insight on this question has been achieved by relying heavily on sophisticated group theoretic techniques.[3] However, it would be nice to have a discussion which embodies directly the tools of S-matrix theory, such as analyticity and factorization of residues. It is the purpose of this note to provide such a discussion and to prove that the daughters deduced from analyticity in unequal-mass scattering give rise to a single Lorentz pole in equal-mass channels to which they couple.

Recently, it was shown that the properties of daughter trajectories which follow from broken (by $u\neq 0$) O(4) symmetry can be derived entirely from analyticity.[4] This could be accomplished by considering unequal-mass scattering because the properties of trajectory functions must be independent of the particular channels to which they couple. It was suggested at that time that it would be difficult to show by taking an equal-mass limit that the analyticity daughters correspond to a single Lorentz pole. Nothing in that discussion could eliminate the possibility that the daughters represented a string of integer-spaced Lorentz poles rather than a single Lorentz pole. The stronger results of the present paper come about by considering, in addition, an amplitude which couples equal- and unequal-mass channels, and by enforcing factorization of Regge-pole residues.

The following development assumes that all particles are spinless and deals with analyticity and factorization constraints enforced simultaneously on two u-channel amplitudes having a common unequal-mass channel. The first amplitude corresponds to the unequal-mass elastic process $M+\mu\rightarrow M+\mu$ designated UU. The second amplitude involves the same unequal-mass channel scattering into an equal-mass channel, $M+\mu\rightarrow m+m$, designated UE.

For the UU amplitude a leading Regge pole which has the value α_0 at $u=0$ must, in order to preserve analyticity at $u=0$, be accompanied by daughters located at α_0-k ($k=1,2,3,\cdots$) at $u=0$. The most singular parts of the daughter residues are calculated in Ref. 4:

$$\gamma_k^{\mathrm{UU}}=\gamma_0^{\mathrm{UU}}\frac{(-1)^k\epsilon^k}{k\,!}\left[\frac{\Gamma(\alpha_0+1)}{\Gamma(\alpha_0-k+1)}\right]^2\times\frac{\Gamma(2\alpha_0-2k+2)}{\Gamma(2\alpha_0-k+2)},\qquad(1)$$

where $\epsilon=(M^2-\mu^2)^2$. The subscript k labels the daughter trajectories and $\gamma_k^{\mathrm{UU}}/u^k$ is the coefficient of the leading power s^{α_0-k} of the kth daughter.

We now proceed to derive the relation analogous to (1) for the amplitude UE by enforcing analyticity at $u=0$. The parent trajectory $\alpha_0(u)$ gives a contribution proportional to $Q_{-1-\alpha_0(u)}(-z_u)$ which for this mass configuration is

$$z_u=\frac{2s+u-\Sigma}{4q_{\mathrm{U}}q_{\mathrm{E}}},\qquad(2)$$

where

$$\Sigma=M^2+\mu^2+2m^2,$$
$$q_{\mathrm{U}}^2=\frac{[u-(M+\mu)^2][u-(M-\mu)^2]}{4u},$$
$$q_{\mathrm{E}}^2=\frac{u}{4}-m^2.\qquad(3)$$

Writing $Q_{-1-\alpha_0}$ in terms of a hypergeometric function, the contribution of the α_0 trajectory to the UE amplitude can be written as

$$T_0^{\mathrm{UE}}=\gamma_0^{\mathrm{UE}}(u)X^{\alpha_0(u)}F(-\tfrac{1}{2}\alpha_0(u),\,-\tfrac{1}{2}\alpha_0(u)+\tfrac{1}{2};\;-\alpha_0(u)+\tfrac{1}{2};\,(2q_{\mathrm{U}}q_{\mathrm{E}}/X)^2),$$
$$X=s+\tfrac{1}{2}u-\tfrac{1}{2}\Sigma.\qquad(4)$$

The need for daughters arises in Eq. (4) because the coefficient of each power of X must be analytic at $u=0$. The expansion of F in Eq. (4) shows that the singular parts of these coefficients are of the form

$$X^{\alpha_0(u)-2n}(q_{\mathrm{U}})^{2n}\propto X^{\alpha_0(u)-2n}/u^n.\qquad(5)$$

Thus, Eq. (5) shows the necessity of daughters at $\alpha_k(0)=\alpha_0(0)-k$, $k=2n$ ($n=1,2,\cdots$), with residues γ_k^{UE} having the behavior $u^{-k/2}$ as $u\rightarrow 0$. We see here that only the even daughters contribute to the UE amplitude, a fact which anticipates the known O(4) result that only even daughters result from a single Lorentz pole in equal-mass scattering. The full amplitude T^{UE} is thus a sum over the parent and daughter trajectories:

$$T^{\mathrm{UE}}(s,u)=\sum_{n=0}^{\infty}T_{2n}^{\mathrm{UE}}(s,u).\qquad(6)$$

The T_k^{UE} are the same as (4) with α_0 and γ_0^{UE} replaced by α_k and γ_k^{UE}. A calculation requiring that the coefficient of $X^{\alpha_0(0)-2n}/u^n$ in Eq. (6) vanish leads in analogy to the similar calculation in Ref. 4 to the following result:

$$\gamma_k^{\mathrm{UE}}=\left(-\frac{m^2\epsilon}{4}\right)^{\frac{1}{2}k}\frac{\pi\sec\pi\alpha_0}{\Gamma(-\alpha_0)}\frac{\Gamma(k-\alpha_0)(\alpha_0-k+\frac{1}{2})\gamma_0^{\mathrm{UE}}}{(\frac{1}{2}k)\,!\,\Gamma(k-\alpha_0+\frac{1}{2})\Gamma(\alpha_0-\frac{1}{2}k+\frac{3}{2})},\quad k=2n\ (n=1,2,3,\cdots).\qquad(7)$$

Assuming that the residues given by Eqs. (7) and (1) factorize, we have the relation

$$(\gamma_k^{\mathrm{UE}})^2 = \gamma_k^{\mathrm{UU}} \gamma_k^{\mathrm{EE}}, \tag{8}$$

where γ_k^{EE} represents the analogous Regge-pole residue in the process $m+m \to m+m$, designated EE. Combining Eqs. (7) and (1) we obtain the following formula for the residues of the equal-mass amplitude EE:

$$\gamma_k^{\mathrm{EE}} = \gamma_0^{\mathrm{EE}} \left(\frac{m^2}{4}\right)^k \frac{k!}{[\frac{1}{2}k]!} \frac{\pi^2 \sec^2 \pi\alpha_0 (\alpha_0 - k + \frac{1}{2})^2 \Gamma(2\alpha_0 - k + 2)}{\Gamma(2\alpha_0 - 2k + 2)[\Gamma(\alpha_0 - \frac{1}{2}k + \frac{3}{2})\Gamma(k - \alpha_0 + \frac{1}{2})]^2}, \quad k = 2n \ (n = 1, 2, 3, \cdots). \tag{9}$$

The contribution of all the even daughters to the EE amplitude is now given by

$$T^{\mathrm{EE}}(u, z_u) = \sum_{n=0}^{\infty} \gamma_{2n}^{\mathrm{EE}}(u)(2q_{\mathrm{E}}^2 z_u)^{\alpha_{2n}(u)} F(-\tfrac{1}{2}\alpha_{2n}(u), -\tfrac{1}{2}\alpha_{2n}(u) + \tfrac{1}{2}; -\alpha_{2n}(u) + \tfrac{1}{2}; z_u^{-2}). \tag{10}$$

At $u=0$ the γ_k^{EE} are given by Eq. (9), and then it can be shown that Eq. (10) is proportional to the expansion of a Gegenbauer function in terms of Legendre functions of the second kind. Thus

$$T^{\mathrm{EE}}(0, z_u) = \mathrm{const}\, D_{-\alpha_0(0)-2}^{\;1}(-z_u), \tag{11}$$

where $D_{-\alpha-2}^{\;1}$ is a Gegenbauer function. The function $D_{-\alpha(0)-2}^{\;1}$, however, corresponds to a single Lorentz pole at $n = \alpha(0)$ since the Gegenbauer functions are the O(4) analogs of the Legendre functions. This completes the demonstration that the daughter Regge trajectories implied by analyticity constitute a single Lorentz pole in equal-mass scattering.

The very powerful role played by the assumption of factorization of residues in this argument should perhaps be re-emphasized. It is factorization which insures that we are dealing with a single Lorentz pole. Without this assumption, it is very doubtful that one can prove that analyticity daughters give rise to a single Lorentz pole, as indicated in Ref. 4.

We remark that in Ref. 4 it is shown that the derivatives of the daughter trajectory functions in UU scattering satisfy certain "mass formulas."

We find exactly the same formulas when the corresponding calculations are performed for the UE process, except of course, the formulas for the odd daughters cannot be derived by considering the UE process. If we had not found the same formulas, there would be a serious inconsistency between analyticity at $u=0$ and Regge theory.

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