S-WAVE $\pi\pi$ SCATTERING LENGTHS FROM SUM RULES AT THE SYMMETRY POINT*

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A sum rule for the $I=2$ amplitude at the symmetry point, $s=t=u=\frac{4}{3}m_\pi^2$, is written down with the assumption that the amplitude satisfies an unsubtracted fixed-energy dispersion relation. This sum rule can be saturated by the S waves and the ρ and f^0 mesons. The known crossing relations at the symmetry point connecting the value and the derivative of the $I=0$ amplitude to those of the $I=2$ amplitude are then used to show that the $I=0$ S-wave scattering length must be small and negative.

In low-energy $\pi\pi$ scattering the $I=1$ P wave and the $I=0$ D wave are quite well understood in terms of the ρ and the f^0 mesons, respectively. On the other hand, no firm experimental information is, as yet, available for the S waves. A number of theoretical calculations regarding S waves have appeared with various conclusions.[1-3] In the following we present a calculation wherein a sum rule for the $I=2$ amplitude is written down at the symmetry point, $s=t=u=\frac{4}{3}m_\pi^2$. Under the assumption that the $I=2$ amplitude for fixed energy satisfies an unsubtracted dispersion relation, this sum can be saturated by low partial waves, i.e., by the S waves and the ρ and the f^0 mesons. The two known crossing relations[1] which connect the value and the derivative of the $I=0$ amplitude to those of the $I=2$ amplitude at the symmetry point are then used to obtain information about the $I=0$ S wave. We find that $\partial A_0{}^0/\partial t$ is positive at the symmetry point, $A_l{}^I(t)$ being the lth partial-wave amplitude for isospin I. If we further assume that the effective-range approximation is good at least in the low-energy region then we find that the only solution which is compatible with the sum rule is the one for which the $I=0$ S-wave scattering length is small and negative.

For the given isospin I in the t channel, the following fixed-t (energy) dispersion relation can be written down for the $\pi\pi$ amplitude $A^I(t,V)$ with possible subtractions (we take $\hbar=c=m_\pi=1$):

$$A^I(t,V)=\frac{1}{\pi}\int_{V_0}^{\infty}dV'A_V{}^I(t,V')\times\left[\frac{1}{V'-V}+(-)^I\frac{1}{V'+V}\right], \quad (1)$$

where $V=\frac{1}{2}(s-u)$ $V_0=2+\frac{1}{2}t$.

We assume that the $I=2$ amplitude satisfies an unsubtracted dispersion relation. This assumption is motivated by the fact that no Regge trajectory, $\alpha_i{}^I(t)$, for $I=2$ is observed with $\alpha_i{}^I>0$.

For $V=0$ the relation for $I=2$ reduces to

$$A^2(t,0)=\frac{2}{\pi}\int_{V_0}^{N}\frac{dV'}{V'}A_V{}^2(t,V') \quad (2)$$

$$=\frac{2}{\pi}\int_{V}^{N}\frac{dV'}{V'}[\tfrac{1}{3}\mathrm{Im}A^0(V',t)-\tfrac{1}{2}\mathrm{Im}A^1(V',T)+\tfrac{1}{6}\mathrm{Im}A^2(V',t)], \quad (3)$$

where we have used crossing relations for the absorptive parts and where N is the value beyond which the absorptive part is negligible. We take N above the f^0 mass region and saturate the integral in (3) by the $I=0$ and $I=2$ S waves, the $I=0$ D wave (the f^0 meson) and the $I=1$ P wave (the ρ meson). A semiquantitative estimate of N can be obtained by writing a dispersion relation for the $I=0$ amplitude with the known Regge asymptotic terms corresponding to the P and the P' trajectories subtracted out.[4] Using the parameters of P and P' obtained by Rarita et al.,[4] we find that $N\approx 2.2$ BeV^2. This value is around halfway between the f^0 and the g (the ρ-recurrence) masses. Therefore, it gives us confidence that we can saturate the integral by the S waves, ρ, and f^0. From now on we shall take N in the neighborhood of 2.2 BeV^2.

We now write down the following two relations, obtained from crossing symmetry, at the symmetry point[1] $s=t=u=\frac{4}{3}$, i.e., at $t=\frac{4}{3}$ and $V=0$:

$$A^0(\tfrac{4}{3},0)=\tfrac{5}{2}A^2(\tfrac{4}{3},0), \quad (4)$$

$$\frac{\partial A^0}{\partial t}(\tfrac{4}{3},0)=-2\frac{\partial A^2}{\partial t}(\tfrac{4}{3},0). \quad (5)$$

Since $t=\frac{4}{3}$ is quite close to threshold $t=4$, one can approximate A^0 and A^2 by their S-wave projections, $A_0{}^0$ and $A_0{}^2$, respectively. Note that the next contributor will be the D wave which has a much stronger threshold dependence and should be negligible. We now write down the effective-range expansion

$$A_0^I(t)=\left(\frac{\nu+1}{\nu}\right)^{1/2}e^{i\delta_0^I}\sin\delta_0^I=\left\{\frac{1}{a_I}+\frac{1}{2}r_I\nu+\frac{2}{\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln[\nu^{\frac{1}{2}}+(\nu+1)^{\frac{1}{2}}]-i\left(\frac{\nu}{\nu+1}\right)^{1/2}\right\}^{-1}, \tag{6}$$

where $\nu[=\frac{1}{4}(t-4)]$ is the square of the c.m. momentum. From (4) to (6), the $I=2$ parameters a_2 and r_2 can be determined in terms of a_0 and r_0:

$$1/a_2=2.5/a_0-1.87r_0+1.67, \tag{7}$$

$$r_2=-3.12r_0+2.48, \tag{8}$$

$$A_0^0(\tfrac{4}{3})=(1/a_0-r_0/3+0.56)^{-1}, \tag{9}$$

$$\frac{\partial A_0^0}{\partial t}(\tfrac{4}{3})=\tfrac{1}{4}(\tfrac{1}{2}r_0+0.30)[A_0^0(\tfrac{4}{3})]^2. \tag{10}$$

From Eqs. (3)-(5), (9), and (10) we get

$$A_0^0(\tfrac{4}{3})=(5/3\pi)(S_0+\tfrac{1}{2}S_2-\tfrac{3}{2}\rho+f), \tag{11}$$

$$\frac{\partial A_0^0}{\partial t}\left(\frac{4}{3}\right)=-\frac{4}{3\pi}\left(\frac{\partial S_0}{\partial t}+\frac{1}{2}\frac{\partial S_2}{\partial t}-\frac{3}{2}\frac{\partial\rho}{\partial t}+\frac{\partial f}{\partial t}\right), \tag{12}$$

where the symbols involving S_I ($I=0,2$) correspond to the contributions of the S waves, approximated by (6), to the right-hand side of (3). The symbols involving ρ and f correspond to the contributions of the ρ and the f^0 mesons as given by their Breit-Wigner forms. In the following discussions and in Figs. 1 and 2 we shall refer to the right-hand sides of Eqs. (11) and (12) as B_0^0 and $\partial B_0^0/\partial t$, respectively.

Before discussing the solutions to (11) and (12), we want to stress the following important points which are independent of any details of the S waves. Note that the contributions S_0, S_2, ρ, f, $\partial\rho/\partial t$, and $\partial f/\partial t$ are always positive, while $\partial S_0/\partial t$ and $\partial S_2/\partial t$ are always negative. Furthermore the contributions to B_0^0 and $\partial B_0^0/\partial t$ of the ρ meson dominate over those of the f^0 meson. Therefore, from (12), we have

$$\frac{\partial A_0^0}{\partial t}(\tfrac{4}{3})>0. \tag{13}$$

In the effective-range approximation this means we must have $r_0<0.6$.

In Figs. 1 and 2, we have plotted A_0^0, B_0^0, and their derivatives as a function of a_0 for typical values of r_0 (<0.6) and for $N=2.2$ BeV^2. Since all the integrals involved in B_0^0 and $\partial B_0^0/\partial t$ are strongly convergent, our conclusion does not depend upon the exact value of N chosen. We see that A_0^0 vanishes at $a_0=0$. It has the same sign as a_0 and its magnitude increases as $|a_0|$ increases. Thus for $a_0<0$, B_0^0 intersects A_0^0 at small values of $|a_0|$ ($a_0>-0.4$). For $a_0>0$, however, the only way we can get a solution to (11) is for the S-wave contribution to become so large as to overcome the negative contribution of ρ. It is found in this case that a_0 has to be very large ($a_0\sim 1.8$) and r_0 negative (~ -0.5). This corresponds to an S-wave resonance around $\nu=1$. Now

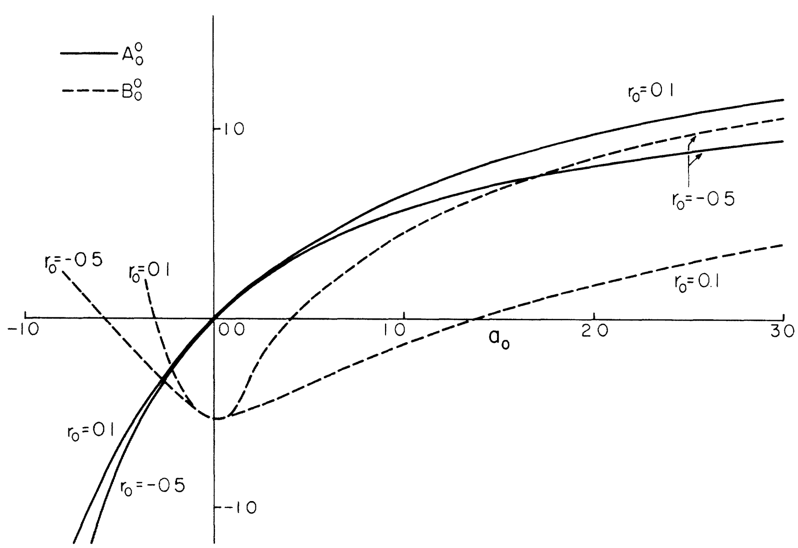

FIG. 1. A plot of A_0^0 vs B_0^0 for $r_0=0.1m_\pi^{-2}$ and $r_0=-0.5m_\pi^{-2}$. Here B_0^0 stands for the right-hand side of expression (11) in the text.

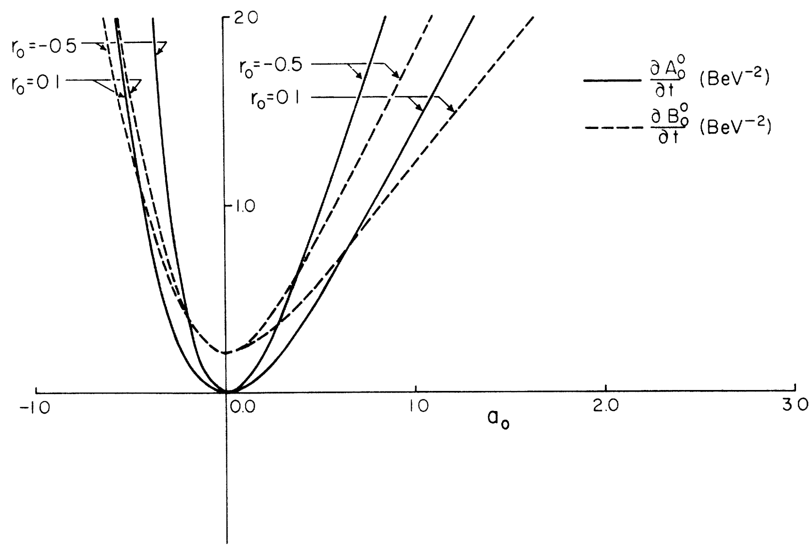

FIG. 2. A plot of $\partial A_0^0/\partial t$ (BeV^{-2}) vs $\partial B_0^0/\partial t$ (BeV^{-2}) for $r_0=0.1m_\pi^{-2}$ and $r_0=-0.5m_\pi^{-2}$. Here $\partial B_0^0/\partial t$ stands for the right-hand side of expression (12) in the text.

let us look at the derivatives. Firstly, $\partial A_0^0/\partial t$ also vanishes at $a_0=0$. It is positive and, unless r_0 is positive, it increases rapidly as $|a_0|$ increases for small $|a_0|$ [see Eq. (10)]. It intersects $\partial B_0^0/\partial t$ at small $|a_0|$. Thus when we consider both the value and derivative conditions we have quite different situations for the cases $a_0>0$ and $a_0<0$.

(i) Positive scattering length ($a_0>0$).—The value condition, (11), demands a large a_0 and small negative r_0. But for negative r_0 the derivative condition, (12), demands a small a_0. Therefore, there can be no simultaneous solution to (11) and (12) for $a_0>0$.

(ii) Negative scattering length ($a_0<0$).—Here both (11) and (12) demand small a_0. We find that a solution to both (11) and (12) exists in the following region:

$$-0.4<a_0<0,$$

$$-0.5<r_0<0.5.$$

We would like to emphasize that our assumption of the effective-range expansion is crucial only insofar as it is used to relate the amplitude at the symmetry point $\nu=-\frac{2}{3}$ to that at threshold, $\nu=0$.[5] As long as the effective-range expansion holds up to $\nu=-\frac{2}{3}$, it should also be good up to roughly the same distance in the positive-ν region. Since the integrals involved in S_I and $\partial S_I/\partial t$ are strongly convergent, no significant change will be caused by possible modification of the expansion above $\nu=1$.

The main assumption we make in our calculation is that the $I=2$ amplitude falls rapidly so that it satisfies an unsubtracted, fixed-energy dispersion relation. This is certainly very plausible, since no $I=2$ Regge trajectory has been observed. Our conclusion is that the $I=0$ S-wave scattering length has to be small and negative. This result is consistent with Chew's conjecture[6] of the $I=0$ phase shift starting from π at threshold and going down.

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[1]G. F. Chew and S. Mandelstam, Phys. Rev. 119, 467 (1960), and Nuovo Cimento 19, 752 (1961); B. R. Desai, Phys. Rev. Letters 6, 497 (1961).

[2]H. J. Rothe, Phys. Rev. 140, B1421 (1965). The author used forward dispersion relations for the $I=0$ amplitude. In his calculation the symmetry-point conditions were not satisfied and the effect of S waves in the dispersion integrals were not considered.

[3]S. Weinberg, Phys. Rev. Letters 17, 616 (1966); T. Akiba and K. Kang, Phys. Letters 25B, 35 (1967); J. R. Fulco and D. Y. Wong, Phys. Rev. Letters 19, 1399 (1967); R. Arnowitt, M. H. Friedman, P. Nath, and R. Suitor, Phys. Rev. Letters 20 475 (1968); L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968).

[4]W. Rarita, R. Riddell, C. Chiu, and R. Phillips, Phys. Rev. 165, 1615 (1968).

[5]It should be mentioned that the effective-range expansion (6) excludes the possibility of any zeros in the S-wave amplitudes. On the other hand, current-algebra considerations do require zeros. We have considered parametrizations which include zeros {e. g., $[\nu/(\nu+1)]^{1/2}\cot\delta = 1/(a-b\nu)+\frac{1}{2}r\nu+(2/\pi)[\nu/(\nu+1)]^{1/2}\ln[\nu^{1/2}+(\nu+1)^{1/2}]$, and we have not been able to find a positive scattering-length solution. In particular, the recent calculation of Brown and Goble (Ref. 3) which uses the current-algebra estimates of Weinberg (Ref. 3) for the scattering lengths and the positions of zeros does not satisfy our sum rules. Their values for A_0^0 and $\partial A_0^0/\partial t$ are 0.046 and 2.86 BeV^{-2}, respectively, while for B_0^0 and $\partial B_0^0/\partial t$ they are 0.26 and 0.86 BeV^{-2}, respectively. We have also compared the recent calculations of E. P. Tryon [Phys. Rev. Letters 20, 769 (1968)] with our sum rules. For the three cases considered by him, the values for A_0^0 and $\partial A_0^0/\partial t$ are (0.018, −0.042, −0.10) and (2.5, 2.5, 2.5) BeV^{-2}, respectively, while B_0^0 and $\partial B_0^0/\partial t$ are $\simeq 0.80$ and $\simeq 1.5$ BeV^{-2}, respectively.

[6]G. F. Chew, Phys. Rev. Letters 16, 60 (1966).

ERRATA

GHOST-ELIMINATING MECHANISM AND TRAJECTORY PARAMETER OF A_2 BY PHOTOPRODUCTION SUM RULE. Shu-Yuan Chu and D. P. Roy [Phys. Rev. Letters 20, 958 (1968)].

Because of a programming error, the Born term was underestimated by a factor of $M^2(=0.88)$ The Born term for $S_0(t=0)$ in Table I should be 5.0 instead of 4.44. Thus the curve S_0 in Fig. 1 should be raised by 0.5 GeV^{-1}. This makes S_0 vanish at -0.7 GeV^2 (exactly at the S_2 minimum) instead of -0.5 GeV^2, and reduces $\alpha(0)$ from 0.6 to 0.45. Therefore our conclusions remain unaltered. We are grateful to Professor K. V. Vasavada for pointing out this error.

SEMICLASSICAL TRANSPORT THEORY IN STRONG MAGNETIC FIELDS. H. F. Budd [Phys. Rev. Letters 20, 1099 (1968)].

The following misprints should be rectified:

Third line: "··· strong magnetic fields, $\omega_c\bar{\tau} \gg 1$ (ω_c is ··· ."

Second line after Eq. (5): "··· and $\bar{v}_y = -E/B \equiv v_d$··· ."

Equation (11):

$$\cdots\left[\hat{C}\left(f_0 - p_y v_d \frac{\partial f_0}{\partial\epsilon}\right)\right]\cdots.$$

PHOTOEMISSION OF ELECTRONS FROM ALKALI AND ALKALINE-EARTH METAL CONTACTS INTO ANTHRACENE. A. Many, J. Levinson, and I. Teucher [Phys. Rev. Letters 20, 1161 (1968)].

After submission of this Letter we became aware that the interpretation of our data was incorrect. It was observed that after a spectral yield curve [Fig. 1(a)] had been taken, the photocurrent measured in the reverse polarity (alkali metal positive) exhibited a spectral response of almost identical form. Furthermore, injection of electrons from the alkali metal in the dark was sufficient to give rise to such "reverse" photocurrent. It therefore follows that optical excitation from traps rather than direct photoemission from the metal contact is responsible for the data in Fig. 1(a).

The reverse photocurrent decays with time of illumination as previously trapped electrons are being photoexcited into the conduction band. That essentially only one discrete set of traps (~0.9 eV deep) is involved in this process has been established by the observation that bleaching by light of 1.4 μ (or in fact by monochromatic light of any other wavelength in the range 1.4-0.44 μ) lowers the entire spectral yield curve by practically the same factor. In principle, the data in Fig. 1(a) can result from direct photoexcitation from this discrete set of traps into various sub-bands of the conduction band. However, as was also pointed out by Mehl and by Castro,[1] the fact that the positions of the four peaks in the range 1.8-2.4 eV coincide with those of the main triplet absorption peaks of anthracene[2,3] strongly indicates that, in this range, detrapping by interaction with triplet excitons is the dominant mechanism. Details will be published elsewhere.

[1]W. Mehl and G. Castro, private communications.

[2]P. Avakian et al., J. Chem. Phys. 29, 1127 (1963).

[3]J. H. Sharp and W. G. Schneider, J. Chem. Phys. 41, 3657 (1964).