Bois, Bell System Tech. J. 38, 271 (1959).
[8]A. M. Portis, Phys. Rev. 91, 1071 (1953).
[9]P. Zeegers and R. van Steenwinkle, Physica 33, 332 (1967).

# PARAMETRIC EXCITATION FROM THERMAL FLUCTUATION AT PLASMA-DRIFT WAVE FREQUENCIES*

Alfred Y. Wong, Martin V. Goldman, Francis Hai, and Richard Rowberg
Department of Physics, University of California, Los Angeles, California
(Received 27 May 1968)

We present experimental verification of the parametric enhancement of thermal fluctuations, the power balance, and the parametric threshold.

We wish to report observation of steady-state parametric enhancement of fluctuations at plasma-drift mode frequencies, leading to eventual coherence and saturation effects. We obtain quantitative experimental agreement with the general theoretical predictions[1-3] valid for steady-state parametric phenomena in any system with Bose-like excitations. The nonlinear parametric enhancement of the fluctuation spectrum may be regarded as a tool for magnifying the details of the original (linear) fluctuation spectrum of a system in a nonequilibrium steady state. Suppose an external oscillator supplies input power $P_0^{\text{ext}}$ at frequency $\omega_0$ to such a system, exciting a coherent mode with frequency $\omega_0$ and amplitude $\varphi_0$ which we shall call the "pump." If there exists a unique or limited number of fluctuating lower-frequency mode pairs $\varphi_{\text{fl}}(\omega_1)$ and $\varphi_{\text{fl}}(\omega_2)$ such that $\omega_1+\omega_2 \simeq \omega_0$,[4] the spectrum of thermal fluctuations

$$S(\omega)=\lim_{T\to\infty}\langle|\varphi_{\text{fl}}(\omega)^2|\rangle_T$$

may be greatly sharpened near $\omega_1$ and $\omega_2$, and the integrated power around these frequencies greatly increased. The line shape of $S(\omega)$ in the vicinities of $\omega_1$ and $\omega_2$ as a function of $|\varphi_0|^2$ has been derived in general for two lossy oscillators coupled parametrically[1] and for the electrostatic modes associated with density fluctuations in a plasma.[2,3] In addition, $|\varphi_0|^2$ saturates as a function of $P_0^{\text{ext}}$ because of channeling of energy into the enhanced fluctuations around $\omega_1$ and $\omega_2$.[3] Before presenting the experimental data we derive expressions for the enhancement and the nonlinear saturation, applicable to the experimental results reported in this paper.

As shown by Goldman[2] and DuBois and Goldman[3] for electrostatic modes, the nonlinear Poisson equation determines the fluctuating potential $\varphi_{\text{fl}}(k_1,\omega)$, where $\omega$ is close to $\omega_1$ and $|\omega_0-\omega|$ is close to $\omega_2$. Poisson's equation takes the form of two coupled equations and describes the parametric mode coupling between any three electrostatic modes (e.g. Langmuir modes, ion-acoustic modes, drift waves, etc.), one of which (the pump) is coherent:

$$\epsilon(\vec{k}_1,\omega)\Phi_{\text{fl}}(\vec{k}_1,\omega)+\Phi_0 M\Phi_{\text{fl}}(\vec{k}_1-\vec{k}_0,\omega-\omega_0)=\frac{4\pi e\rho_{\text{fl}}(\vec{k}_1,\omega)}{\theta k_1^2},$$

$$\epsilon(\vec{k}_1-\vec{k}_0,\omega-\omega_0)\Phi_{\text{fl}}(\vec{k}_1-\vec{k}_0,\omega-\omega_0)+\Phi_0 M^*\Phi_{\text{fl}}(\vec{k}_1,\omega)=\frac{4\pi e\rho_{\text{fl}}(\vec{k}_1-\vec{k}_0,\omega-\omega_0)}{\theta|\vec{k}_1-\vec{k}_0|^2},$$

$$\Phi_{\text{fl}}(\vec{k},\omega)=e\varphi_{\text{fl}}(\vec{k},\omega)/\theta, \tag{1}$$

where $\rho_{\text{fl}}(\vec{k},\omega)$ is a thermally fluctuating charge density source, $\epsilon(\vec{k},\omega)$ is the linear longitudinal dielectric function whose zeros give the electrostatic collective modes at $\omega_1$ and $\omega_2$, $M$ is the dimensionless mode-coupling matrix element, $\theta$ is the common temperature of electrons and ions, and $\Phi_0=e\varphi_0/\theta$, where $\varphi_0$ is the real amplitude of the coherent pump.

For a stable system in a nonequilibrium steady state, with mode coupling ignored, we may use a formal extension of the fluctuation-dissipation theorem,

$$\lim_{\Omega T\to\infty}\frac{\langle|\rho_{\text{fl}}(\vec{k},\omega)|^2\rangle}{\Omega T}=\frac{k^2\theta f(\vec{k},\omega)\,\mathrm{Im}\,\epsilon(\vec{k},\omega)}{2\pi\omega},$$

where $\Omega$ is the (large) volume of the system, and $f(k,\omega)$ is some nonresonant function of $\omega$, of $k$, and of parameters describing the nonequilibrium configuration (such as density or temperature gradients). For an equilibrium system, $f=1$. As shown by Goldman,[2] for lossy mode coupling ($M$ complex), $f$ may have a weak dependence on $\varphi_0$.

In the presence of mode coupling, the solution of Eq. (1) is best exhibited by specializing to the case of experimental concern, $\omega_1=\omega_2=$ the fundamental frequency in a system with finite geometry. We also assume that $\vec{k}_1$ is the fundamental wave number and that the wave number of the pump is $\vec{k}_0=2\vec{k}_1$, and use the acousticlike dispersion relation for drift waves, $\omega_1(k_1)\simeq\vec{k}_1\cdot\vec{V}_D$ where $\vec{V}_D$ is the electron diamagnetic drift velocity. This implies[4] $\omega_0(k_0)\simeq 2\omega_1(k_1)$. In the neighborhood of a resonance, $\omega_1$, the drift-wave dielectric function is

$$\epsilon(k_1,\omega)\simeq\frac{k_D^{\,2}}{k_1^{\,2}}\frac{[\omega-\omega_1(k_1)+i\gamma_1(k_1)]}{\omega_1(k_1)},$$

where $k_D^{\,2}=4\pi ne^2/\theta$ is the Debye wave number squared. $\gamma_1$ is a positive relaxation rate (achieved experimentally by end-plate losses and ion-ion collisions). With the above approximations, we find a line shape not very different from that of DuBois and Goldman[3] (for parametrically coupled Langmuir and ion-acoustic waves satisfying $\omega_1\gg\omega_2$ and $\gamma_1\gg\gamma_2$):

$$S(\omega)=\lim_{\Omega,\,T\to\infty}\frac{(\Delta^3k_1)}{(2\pi)^3}\frac{\langle|\Phi_{\mathrm{fl}}(k_1,\omega)|^2\rangle}{\Omega T}$$
$$=\frac{\Delta^3k_1}{(2\pi)^3}\frac{2\gamma_1 f(k_1,\omega)}{n}\frac{(\omega-\omega_1)^2+\gamma_1^{\,2}(1+\varphi_0^{\,2}/\varphi_{\mathrm{crit}}^{\,2})}{[(\omega-\omega_1)^2+\gamma_1^{\,2}(1+\varphi_0/\varphi_{\mathrm{crit}})^2][(\omega-\omega_1)^2+\gamma_1^{\,2}(1-\varphi_0/\varphi_{\mathrm{crit}})^2]},\qquad(2)$$

where $\omega_1=|\omega_1(k_1)|$ and

$$\frac{k_1^{\,2}}{k_D^{\,2}}|M|\frac{e\varphi_{\mathrm{crit}}}{\theta}=\frac{\gamma_1}{\omega_1}\qquad(3)$$

defines the threshold $e\varphi_{\mathrm{crit}}/\theta$. $(k_1^{\,2}/k_D^{\,2})|M|$ will be called the coupling coefficient.

We wish to point out an important application of the parametric excitation by means of Eq. (2): The introduction of the pump $\varphi_0$ serves to amplify any form factor $f(k_1,\omega)$ that may be present in the original (linear) fluctuation spectrum and makes it possible to study the original spectrum in the equilibrium and nonequilibrium regions. The forms for the line shape given here and in Ref. 3 are fairly general and not restricted to electrostatic modes. The enhanced peak and the integrated spectrum around $\omega_1$ are easily related to the $\varphi_0=0$ values,

$$S(\omega_1)=S_{M=0}(\omega_1)\frac{[1+(\varphi_0^{\,2}/\varphi_{\mathrm{crit}}^{\,2})]}{[1-(\varphi_0^{\,2}/\varphi_{\mathrm{crit}}^{\,2})]^2},\qquad(4)$$

$$\int\frac{d\omega}{2\pi}S(\omega)=\frac{[1+(\varphi_0^{\,2}/\varphi_{\mathrm{crit}}^{\,2})]}{[1+(\varphi_0/\varphi_{\mathrm{crit}})][1-(\varphi_0^{\,2}/\varphi_{\mathrm{crit}}^{\,2})]}\times\int\frac{d\omega}{2\pi}S_{M=0}(\omega),\qquad(5)$$

where $S_{M=0}$ is the assumed linear fluctuation spectrum without mode coupling. Equations (4) and (5) have been derived assuming the pump amplitude $\varphi_0$ independent of the enhancement of the fluctuating spectrum. Since a finite power $P_0^{\,\mathrm{ext}}$ is supplied to the system, $\varphi_0$ must always remain below $\varphi_{\mathrm{crit}}$ to conserve energy. Quantitatively, $P_0^{\,\mathrm{ext}}$ must equal the power dissipated at $\omega_0$ due to the linear dissipation $\gamma_0$ plus the excess over equilibrium of the incoherent power dissipated at $\omega_1$ by the linear dissipation $\gamma_1$[3]:

$$\frac{P_0^{\,\mathrm{ext}}}{\Omega}=C_1\varphi_0^{\,2}+C_2\int\frac{d\omega}{2\pi}[S(\omega)-S_{M=0}(\omega)],\qquad(6)$$

where

$$C_1=2k_D^{\,2}\gamma_0\ \text{and}\ C_2=2k_D^{\,2}\gamma_1(\theta^2/e^2).$$

Equation (6) indicates that the pump power must change as a result of the enhancement of the fluctuating spectrum around the signal frequency.

Our experiment was conducted in a highly ionized cesium or potassium plasma of a $Q$ device.[5] The plasma is generated by contact ionization of a neutral alkali beam by a hot tungsten plate at each end of the column and is confined radially by a uniform axial magnetic field. The neutral beam is adjusted to obtain a density-gradient maximum near the center of the column. A temperature- and a potential-gradient maximum occur near the plate edge. The resulting plasma has a diameter of 5 cm (distance between 30% points) and a temperature gradient of approximately 90 deg/cm monotonically decreasing from the center toward the edge. Our investigation of parametric coupling is mainly concerned with drift waves occurring near the density-gradient maximum although similar parametric processes also take place for waves occurring near the plate edge. In a cylindrical plasma, these waves are essentially ion-acoustic waves propagating azimuthally, almost perpendicular to the axial magnetic field ($k_\varphi \gg k_\parallel$), the periodicity of the wave in the azimuthal direction giving rise to resonant modes with integral relationship $k_\varphi = m/r_0$ where $m$ = integral azimuthal mode number and $r_0$ = radius at which the mode is observed. Plasma drift waves are chosen for the parametric experiments because their damping rates could be varied arbitrarily close to the marginally stable limit ($\gamma \ll \omega$) through changes in the magnetic field, density, and the column length; secondly, their linear dispersion $\omega \approx \vec{k} \cdot \vec{V}_{De}$ easily satisfies the conservation of wave numbers and frequencies as required by parametric processes[4]:

$$\omega_0 \simeq \omega_1 + \omega_2, \quad \vec{k}_0 \simeq \vec{k}_1 + \vec{k}_2.$$

Excitation of the higher mode (the pump) was achieved by modulating the potential of either a floating probe or a probe biased at or below the plasma potential.[6] Drift waves were detected by a floating Langmuir probe. Both the excitation and the detection probe were placed at the density gradient maximum. To effect parametric excitation, the excitation probe is driven at the frequency $\omega_0 = 2\omega_1$ corresponding to the $m = 2$ mode. A spectrum analyzer is swept near the $m = 1$ mode to monitor the fluctuation spectrum. Representative power spectra of the floating-potential fluctuations as the pump amplitude $\varphi_0$ is increased are plotted in Fig. 1(a) together with the line shape predicted by Eq. (2) for the equilibrium condition $[f(k, \omega) = 1]$. The threshold $e\varphi_{\text{crit}}/\theta = 0.04$ chosen for the best fit to the line shape is consistent with that determined by other criteria mentioned below. The asymmetry present in the experimental curve could be caused by density and temperature gradients whose effects are contained in $f(k, \omega)$. In the more nonlinear regime above threshold the spectrum about $\omega_1$ narrows considerably (<50 Hz) and phase coherence between the signal and the pump is evident in Fig. 1(b). A higher order of nonlinearity begins to appear with the generation of an $m = 3$ mode possibly due to the interaction of the signal back on the pump. It is unlikely that this enhancement of

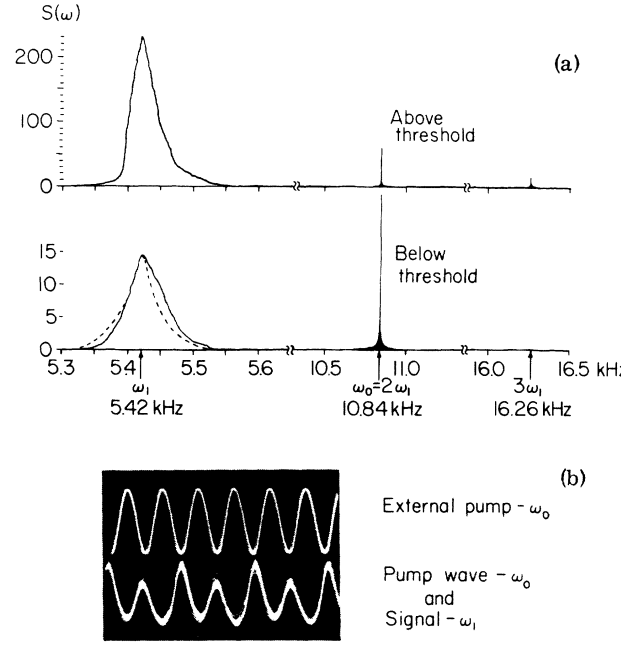


FIG. 1. (a) Representative power spectra illustrating the enhancement from fluctuation about 5.42 kHz as the pump power at 10.84 kHz is increased. The theoretical curve (dashed) is obtained from Eq. (2) with $\gamma_1 = 1 \times 10^3$ and $\varphi_0/\varphi_{\text{crit}} = 0.85$. The ordinate has been expanded in the lower trace. (b) Oscillogram showing the signal on the excitation grid (upper trace) and the wave detected by the Langmuir probe near threshold. The lower trace representing the simultaneous occurrence of the self-consistent pump and the excited coherent signal can be Fourier decomposed as $\varphi_0 \cos(2\omega_1 t) + \varphi_1 \cos(\omega_1 t + \alpha)$, where $\alpha \simeq 0$. A preliminary analysis similar to that of R. Goldman [University of Maryland Technical Note No. BN-521, 1967 (unpublished)] for three coherent modes has shown that optimum coupling should take place for this condition of nearly zero phase shift.

spectrum could result from changes in zeroth order plasma conditions (e.g., changes in the density gradient or temperature) because the self-consistent pump $e\varphi_0/0$ is always less than 10%.[7] Furthermore the pump frequency must fall within a narrow range (50 Hz) about an optimum frequency and the peak of the enhanced spectrum occurs always at exactly half the pump frequency.

A continuous monitor of both the self-consistent pump power $P_0$ (proportional to $\varphi_0^2$) and the peak fluctuating signal power [proportional to $S(\omega_1)$] as functions of $P_0^{\text{ext}}$ is shown in Fig. 2. The departure of $P_0$ from the initial linear rise coincides with the increase in $S(\omega_1)$ required by power conservation described by Eq. (6). In the

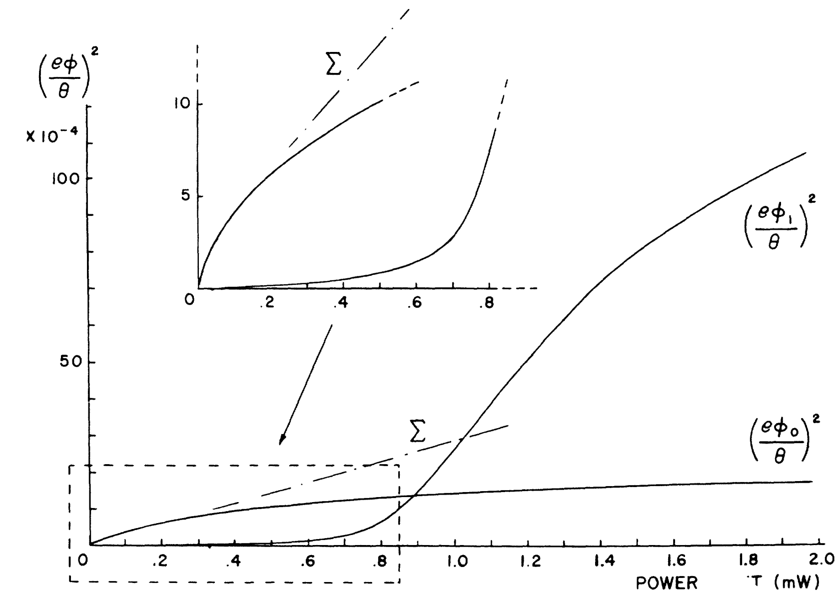


FIG. 2. Peak spectral density $(e\varphi_1/\theta)^2$ at $\omega_1$ and the self-consistent pump wave $(e\varphi_0/\theta)^2$ at $\omega_0$ as functions of the oscillator power $P_0^{\text{ext}}$ which is varied continuously in these measurements. $\Sigma$ represents the total power dissipated at $\omega_0$ and $\omega_1$ obtained by computing the right-hand side of Eq. (6) taking into account the total enhanced spectrum about $\omega_1$.

region where $S(\omega)$, the power spectrum about $\omega_1$, is sufficiently enhanced and can be determined accurately, Eq. (6) is verified by showing that the sum of the dissipated power around $\omega_0$ and $\omega_1$ varies linearly with $P_0^{\text{ext}}$ as shown in Fig. 2.

Replotting the peak spectral density $S(\omega_1)$ as a function of the self-consistent pump power $\varphi_0^2$ in Fig. 3(a) shows that the experimental results can be described theoretically by Eq. (4) provided a threshold of $e\varphi_{\text{crit}}/\theta = 0.04$ is chosen. This

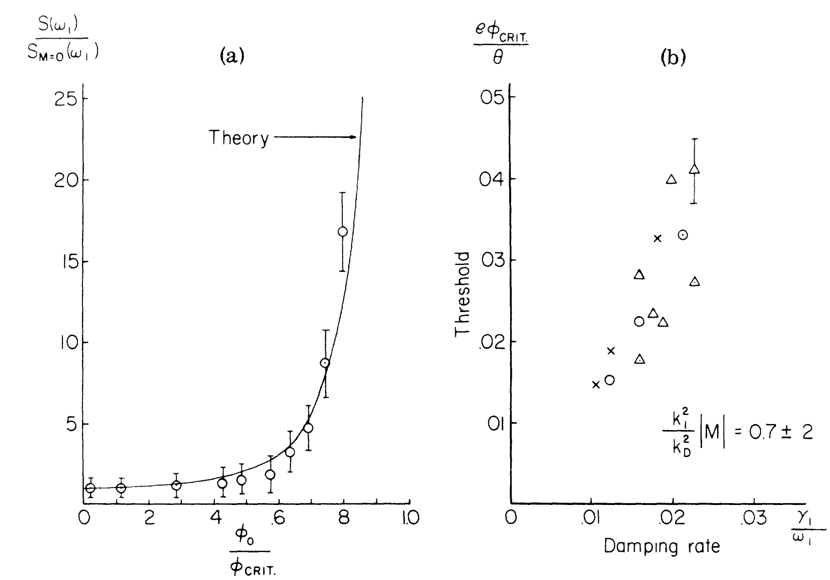


FIG. 3. (a) Peak spectral density $S(\omega_1)$ normalized to the spectral density without mode coupling $S_{M=0}(\omega_1)$ as a function of the normalized pump wave amplitude $\varphi_0/\varphi_{\text{crit}}$. The theoretical curve is described by Eq. (4). The data are for a potassium plasma, $N_0 = 1.5\times10^{10}$ cm$^{-3}$, $T = 2200$°K, $B = 660$ G. The axial column length = 70 cm. (b) Normalized threshold pump wave amplitude $e\varphi_{\text{crit}}/\theta$ as a function of the damping rate of the signal mode. The different symbols represent different runs. The coupling coefficient $(k_1^2/k_D^2)|M|$ is given by the inverse slope of the line best fitted to the experimental data. The ranges of the experimental parameters are $7.0\times10^9$ cm$^{-3} < N_0 < 2\times10^{10}$ cm$^{-3}$ and 650 G $< B <$ 800 G. The data are for a potassium plasma.

threshold agrees within a factor of 2 with the calculated threshold[8] of 0.08 computed from the nonlinear fluid equations describing drift waves. Near threshold where the power spectrum can be accurately measured we have also found a similar though less rapid increase in the integrated spectrum as described by Eq. (5). Thus the threshold for parametric excitation is defined consistently as (1) the value of $e\varphi_0/\theta$ that yields the best fit of the enhancement of $S(\omega)$, $S(\omega_1)$, and $\int_{\omega_1} S(\omega)d\omega$; (2) the value of $e\varphi_0/\theta$ beyond which phase coherence is detectable between the pump and the signal corresponding to a narrowing of the frequency spectrum.

To facilitate parametric excitation under low pump power and to simplify the theoretical treatment, we have adjusted the plasma parameters (magnetic field and density) such that $\gamma_1 \ll \omega_1$. Under these conditions, both the threshold $\varphi_{\rm crit}$ and the damping $\gamma_1$[6] are experimentally determined. The results given in Fig. 3(b) indicate approximately a linear variation and verify Eq. (3) which states that the threshold must increase with the damping rate of the signal mode. The measured coupling coefficient $k_1^2|M|/k_{\rm D}^2$ obtained from the inverse slope is $0.7 \pm 0.2$ compared with a calculated value[8] of 0.25. Considering the plane-wave approximation used in the theoretical calculation this agreement is tolerable.

In summary, our experimental results have demonstrated the following: (1) The enhancement from fluctuations has the theoretically predicted functional form; (2) the saturation of the pump amplitude is in agreement with the power balance between the source, the pump, and the signal; (3) the variation of the threshold with the damping of the signal mode is functionally correct and checks approximately with the theoretically computed coupling coefficient.

Our experiments have demonstrated the importance of parametric coupling for even relatively low-amplitude resonant drift modes in a plasma of finite geometry. Our findings differ considerably from the calculation in an infinite plasma[9] which shows the parametric coupling to be insignificant. The methods used here in identifying the parametric process through observation of the enhancement from fluctuations is quite general and should be applicable to a large class of parametric processes.

We wish to express our gratitude to Mr. G. Weyl for making available his prepublished results to us. We are grateful to Dr. D. DuBois for many enlightening discussions. The help of Mr. Z. Lucky has been invaluable in our experiments.

---

*Research supported by the National Aeronautics and Space Administration under Grant No. NGR 05-087-006.

[1]W. G. Wagner and R. W. Hellwarth, Phys. Rev. 133, A915 (1964).

[2]M. V. Goldman, Ann. Phys. (N.Y.) 38, 95 (1966).

[3]D. F. Dubois and M. V. Goldman, Phys. Rev. 164, 207 (1967).

[4]It is shown in Ref. 3 that this frequency-matching condition need only be met to within the combined linear relaxation rate $\gamma_1+\gamma_2$ (assumed $\ll \omega_1+\omega_2$). Experimentally we find that when the resonant frequencies are not integral multiples of the fundamental because of corrections resulting from finite Larmor radius effects and ion-ion collisions, parametric excitation still occurs if the widths of the resonances are sufficiently large as to allow frequency matching.

[5]N. Rynn and N. D'Angelo, Rev. Sci. Instr. 31, 1396 (1960).

[6]A. Y. Wong and R. Rowberg, Phys. Rev. Letters 18, 390 (1967).

[7]A. Y. Wong, F. Hai, and R. Rowberg [International Symposium on Fluctuation and Diffusion in Plasmas, Princeton University, 1967 (unpublished), paper B-6] reported the distortion of the zero-order density gradient for $e\varphi/\theta > 10\%$ as reflected in their measurements of $\omega_R$ and $\omega_i$ of large-amplitude drift waves.

[8]G. Weyl [University of California, Los Angeles, Plasma Physics Group Report No. R-32, 1968 (unpublished] found the coupling coefficient to be

$$(k_1^2/k_{\rm D}^2)|M| \simeq 8k_1^2\rho_i^2\nu_{ei}k_1V_D/k_z^2a_e^2,$$

where $\rho_i$ is the ion-gyro radius, $\nu_{ei}$ the electron-ion collision frequency, $k_z$ the parallel wave number, and $a_e$ the electron thermal velocity.

[9]T. H. Dupree, Phys. Fluids 10, 1049 (1967). His consideration of a broad spectrum of drift waves permits the frequency-matching condition $\omega_0 = \omega_1+\omega_2$ to be satisfied by an infinite number of frequency pairs $(\omega_1, \omega_2)$.

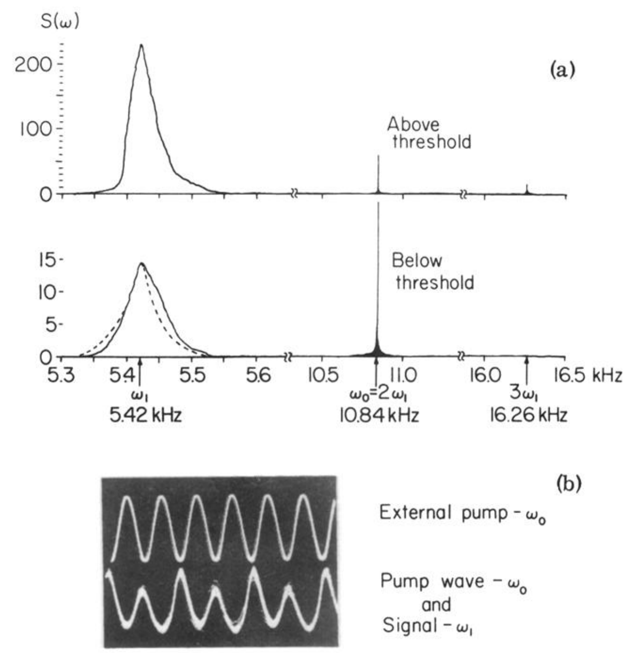


FIG. 1. (a) Representative power spectra illustrating the enhancement from fluctuation about 5.42 kHz as the pump power at 10.84 kHz is increased. The theoretical curve (dashed) is obtained from Eq. (2) with $\gamma_1 = 1\times 10^3$ and $\varphi_0/\varphi_{\rm crit} = 0.85$. The ordinate has been expanded in the lower trace. (b) Oscillogram showing the signal on the excitation grid (upper trace) and the wave detected by the Langmuir probe near threshold. The lower trace representing the simultaneous occurrence of the self-consistent pump and the excited coherent signal can be Fourier decomposed as $\varphi_0 \cos(2\omega_1 t) + \varphi_1 \cos(\omega_1 t + \alpha)$, where $\alpha \simeq 0$. A preliminary analysis similar to that of R. Goldman [University of Maryland Technical Note No. BN-521, 1967 (unpublished)] for three coherent modes has shown that optimum coupling should take place for this condition of nearly zero phase shift.