GENERATION OF PARALLEL DAUGHTERS FROM SUPERCONVERGENCE

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A dynamical model based on superconvergence equations and Regge asymptotics generates parallel daughter trajectories for positive and negative t. The results are found by studying different reactions both by means of sum rules and the partial-wave analysis of Regge terms proposed by Schmid.

In recent papers[1-4] we have found the rather surprising result that a few resonances in the direct channel are able to generate in the crossed channel a rather large piece of meson trajectories with the expected properties needed to explain high-energy scattering. In particular the study of $\pi\pi \to \pi\omega$ [1] has proved to be a remarkable laboratory for the ρ-trajectory bootstrap. The method has by now been extended to a large number of processes like $\pi\pi \to \pi A_2$ [2] and $\pi\pi \to \pi H$,[3] where H is a 1^+, $G=-1$ object. By virtue of their inelastic nature these processes are free from the statistically cumbersome features associated with diffraction peaks. In all of them the same leading trajectories are involved, and the properties demanded of the trajectories in order to satisfy the sum rules are always strikingly consistent with each other, in spite of the different space-time structure and masses of states involved.

While extending the sum rule related to the reaction $\pi\pi \to \pi\omega$ by including higher spin intermediate states for saturation purposes, one finds out that at the 3^- level the equality of the Regge and resonance side is very satisfactory in a large region of t. However, when the 5^-, 7^-, etc. states are included, the resonance side becomes smaller compared with the Regge side.[2] More precisely: One cannot achieve a bootstrap model in which one trajectory is able to sustain itself.

Several possibilities are open at this stage: One can assume (a) that the dynamics is very complicated and that many unknown trajectories are needed, (b) that the continuum becomes important in this region, and (c) that the extra strength needed has a simple and necessary dynamical origin. It is our purpose here to study this last possibility and its connection with Regge "daughter" trajectories.

In our previous work we parametrized the amplitude using $\nu = \frac{1}{4}(s-u)$ as asymptotic variable, a choice that has the proper behavior for the leading terms. Nevertheless other trajectories must exist to cancel the unwanted singularities of the nonleading terms at $t=0$, and most recent analyses based on the Van Hove[5] and Bethe-Salpeter models[6] seem to indicate a very different behavior for these trajectories. Also, experimentally it seems that one would predict unobserved particles if these trajectories were to grow parallel to the leading ones with a spacing $\Delta J = 1, 2, 3, \cdots$. However, if the spacing is $\Delta J = 2, 4, \cdots$, there is no evidence against parallel daughters and in fact some particles could easily be accomodated on these trajectories Since our reaction only couples to trajectories spaced (at $t=0$) by two units of J, we may hope that the $\pi\pi \to \pi\omega$ system might be coupled to an angular momentum set of trajectories as the one depicted in Fig. 1.

If this form of bootstrap is viable, we have to prove that the Regge trajectories contain all these resonances and that reciprocally these resonances can generate the trajectories in the sense of the sum rules. We start by looking into the content of the Regge terms. For this purpose we use the recently proposed technique of Schmid[7] of performing a partial-wave analysis of the Regge amplitude. We consider first the reaction $\pi\pi \to \pi\omega$ for which we have a wealth of information

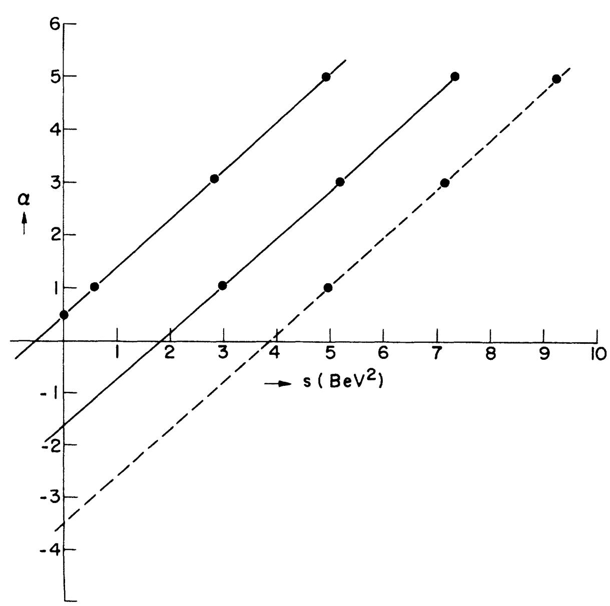

FIG. 1. Trajectory family with $G=+1$ and $I=1$.

derived from the sum rules.[1,2] The invariant amplitude is defined as

$$T_{\pi\pi\to\pi\omega}=\epsilon_{\mu\nu\rho\sigma}p_{1\mu}p_{2\nu}p_{3\rho}e_{\sigma}A(s,t,u), \qquad (1)$$

where p_i are the pion momenta and e_σ is the polarization vector of the ω.

At high ν and fixed t we parametrize A in the Regge form

$$A \xrightarrow[\nu\to\infty]{} \frac{c}{\Gamma(\alpha)}(\nu/\nu_1)^{\alpha(t)-1}\xi(\alpha). \qquad (2)$$
(fixed t)

From our analysis of the sum rules[2] we have found c = constant, $\alpha(t)\simeq\frac{1}{2}+t$, and $\nu_1=1/2\alpha'$ $\approx 0.5\ \mathrm{BeV}^2$. $\xi(\alpha)$ is the signature factor $[1-e^{-i\pi\alpha(t)}]/\sin\pi\alpha(t)$.

The partial-wave projection is given

$$\mathfrak{F}^{J}_{(s)}=\frac{[J(J+1)]^{1/2}}{8(2s)^{1/2}}S_{\pi\pi}S_{\pi\omega}\times\int_{-1}^{+1}d\cos\theta_s A_{\mathrm{Regge}}(s,t,u)\times[P_{J-1}-P_{J+1}], \qquad (3)$$

where A_{Regge} is given by Eq. (2) after symmetrization between t and u. This t,u symmetry makes the forbidden partial waves vanish automatically. Also,

$$S_{ab}=\{[s-(m_a+m_b)^2][s-(m_a-m_b)^2]\}^{1/2}. \qquad (4)$$

These integrals are the ones that generate the Argand-type diagram[7] of our inelastic process.

We find the striking results of Fig. 2(a). The

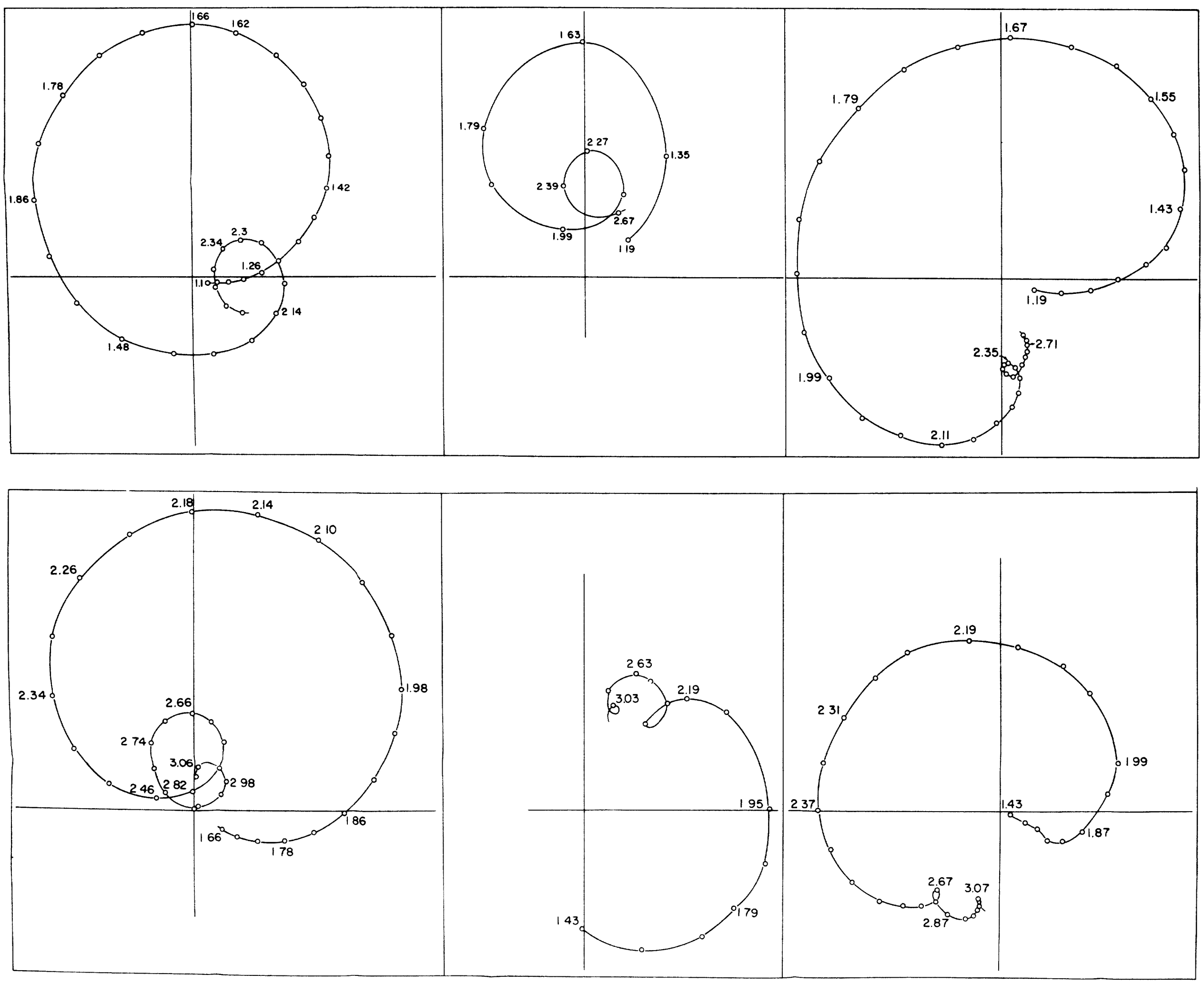

FIG. 2. (a) Argand plots for $l=3$ and $l=5$, reaction $\pi\pi\to\pi\omega$; (b) same plots for $l=3$ and 5, for $\pi\pi\to\pi H$; and (c) same as (b) for the helicity amplitude $\mathfrak{F}_{\lambda=1}{}^{J}$. Numbers along curves are masses in BeV; ordinates and abscissas are Im and Re parts of the amplitudes, respectively.

amplitude turns around more than once generating presumably more than one resonance. Most surprising, the masses of these particles are such that they lie on parallel "daughter" trajectories with $\Delta J = 2$ spacing. The couplings that can be read from the plots are in qualitative agreement with the input and, most rewarding, the background is very small, a fact that provides further support for the idea of saturation by means of sharp resonances in the meson case.[1]

However the most striking property of the resonance, the pole, is lacking in this representation of the amplitude.[7] To get further indication that one is not obtaining some spurious reflections of variations of the amplitude in the t channel, we studied another similar reaction that is controlled by the same trajectory with the same selection rules: $\pi\pi \to \pi H$.[3] In this reaction there are two helicity amplitudes, and the space-time structure, as well as the projection operators involved, is completely different compared with the former case.[3] In fact now, because of parity,

$$T_{\pi\pi\to\pi H} = i(P_2+P_3)_\mu e_\mu A_1(s,t,u) + i(P_2-P_3)_\mu e_\mu B_1(s,t,u). \tag{5}$$

The partial waves are given by the coefficients of the Wigner functions so that we obtain

$$\mathfrak{F}_{\lambda=1}{}^J(s) = \left(\frac{2J(J+1)}{s}\right)^{1/2} S_{\pi\pi}\int_{-1}^{+1} d\cos\theta_s (A_1+B_1)_{\text{Regge}}[P_{J-1}-P_{J+1}], \tag{6}$$

$$\mathfrak{F}_{\lambda=0}{}^J(s) = \frac{1}{4m_H}\left[S_{\pi H}\int_{-1}^{+1} d\cos\theta_s (3A_1-B_1)_{\text{Regge}} P_J + \frac{S_{\pi\pi}}{s}(s+m_H{}^2-m_\pi{}^2)\int_{-1}^{+1} d\cos\theta_s (A_1+B_1)_{\text{Regge}} \cos\theta_s P_J(\cos\theta_s)\right]. \tag{7}$$

A_1+B_1 and $3A_1-B_1$ are amplitudes with definite $u \to t$ crossing. Their high-s parametrization is

$$A_1+B_1 \to \xi(\alpha)[\beta^{A_1}(\nu/\nu_1)^{\alpha(t)-1} + \beta^{B_1}(\nu/\nu_1)^{\alpha(t)}] + (u \to t), \tag{8}$$

$$3A_1-B_1 \to \xi(\alpha)[3\beta^{A_1}(\nu/\nu_1)^{\alpha(t)-1} - \beta^{B_1}(\nu/\nu_1)^{\alpha(t)}] - (u \to t). \tag{9}$$

We still choose $\nu_1 = 1/2\alpha'$ and β^{A_1} and β^{B_1} constant, and let the ratio β^{A_1}/β^{B_1} vary consistently with the procedure of Ref. 3. The Argand plot for $\beta^{A_1}/\beta^{B_1} = -6$ (value obtained in Ref. 3) is given in Figs. 2(b) and 2(c).

It is remarkable that these very different formulas lead to the same plots, concerning position of particles which, as true resonances demand, do appear in all helicity states. We believe these two independent and complementary tests are quite convincing on the point that the Regge leading term contains all this structure.

Since the mathematics of this exercise is rather obscured by the cumbersome integrations, we would like to present an example of what is involved. Consider the $\pi\pi \to \pi\omega$ case. Instead of an angular-momentum projection of $A(s,t,u)$, we develop it in powers of $\nu_{tu} = (t-u)$ (a Khuri expansion):

$$A = \sum_n a(s,n)\nu_{tu}{}^n. \tag{10}$$

From the expression of A_{Regge} we easily obtain ($\Sigma = m_\omega{}^2 + 3m_\pi{}^2$)

$$\text{Im}a(s,0) = \left(\frac{3s-\Sigma}{2\nu_1}\right)^{\frac{1}{2}\alpha'(\Sigma-s)+\alpha(0)-1} \times \frac{1}{\Gamma[\frac{1}{2}\alpha'(\Sigma-s)+\alpha(0)]}, \tag{11}$$

where the maxima and minima of $\text{Im}a(s,0)$ are clearly related to the behavior of $\Gamma^{-1}[\frac{1}{2}\alpha'(\Sigma-s)+\alpha(0)]$. The zeros of this function and the linear argument in s are responsible for the appearance of parallel daughters. Also, a change in the generating trajectory that is displaced by two units reinforces the same extreme points. Hence by feeding back the new trajectories the results are reinforced and the picture is not distorted. This was verified in all other cases as well.

We now rephrase the problem in the sum-rules language. This has three main advantages:
(1) The resonances are inserted as real poles of the scattering amplitudes; there is no problem such as that of discovering the exact meaning of

the pictures in the Argand plot. (2) We obtain a set of algebraic equations and we can look for its mathematical solutions. (3) We can work at fixed t and determine self-consistently the t dependent of the residue function.

On the other hand, the method of Ref. 7 is heuristically more powerful and is able to check other details like the importance of continuum.

Consider $\pi\pi \to \pi\omega$. We want to see whether the insertion of resonances is enough to balance the sum rules and, in turn, whether the conspiracy conditions imposed by analyticity at $t=0$ (for unequal-mass scattering) do yield some information about these trajectories.

The general form of the sum rule is[2]

$$\sum_{i=0}^{n} \frac{\nu_i(2\nu_1)^{-2i}}{(2i)!\alpha'} \bar{P}_{2i+1}'(\cos\theta_s) + r\sum_{j=0}^{n_1} \frac{\nu_j^D(2\nu_1^D)^{-2j}}{(2j)!\alpha_D'} \bar{P}_{2j+1}'(\cos\theta_s) + \cdots = \alpha\Gamma^{-1}(\alpha+2)\bar{\nu}^2(\bar{\nu}/\nu_1)^{\alpha-1} + \cdots. \quad (12)$$

In Eq. (12) $\nu_i = \frac{1}{4}(2m_i^2 + t - \Sigma)$, $r = c_D/c$, where we have used Eq. (2) and a similar one ($c \to c_D$, $\alpha \to \alpha_D$) for the first daughter contribution. Also $\bar{P}_{2i+1}'(\cos\theta_s)$ is proportional to $P_{2i+1}'(\cos\theta_s)$ and has asymptotic behavior

$$P_{2i+1}'(\cos\theta_s) \underset{t\to\infty}{\longrightarrow} t^{2i}.$$

We have assumed linear trajectories in our region of t. The i summation is that of the contribution of the resonances on the parent trajectory, that over j is of the first daughter, and so on. On the right-hand side of Eq. (12) we have written the leading term which will be followed by the nonleading ones (which are regular at $t=0$ after the singularities have been canceled between "parent and daughter" trajectories).

If we restrict ourselves, as an example, to saturation with $\rho + R$ $(3^-, 1680)$ and we neglect other terms in the right-hand side of (12) (we can show they are unimportant), the sum rule reads[2]

$$\nu_\rho \alpha' + \frac{\nu_R \alpha'}{8\nu_1^2}\bar{P}_3' + r\left(\frac{\alpha'}{\alpha_D'}\right)^2 \nu_D = \tfrac{1}{8}\alpha(\alpha+2)(\alpha+3)\Phi(\alpha), \quad (13)$$

where $\Phi[\alpha(t)] = 2[\frac{1}{2}(\alpha+6)]^{\alpha+1}[\Gamma(\alpha+4)]^{-1}$ is a function that is equal to 1 in all the region of interest to a high degree of accuracy.[8]

The conditions implied by Eq. (13) to be satisfied as an algebraic equation demand that $\alpha_D' = \alpha'$ and also fix $\beta^D(m_D^2)/\beta(m_\rho^2) = -1/40$ [remember that $\alpha_D(0) = \alpha(0) - 2$].

The $t=0$ analyticity conditions can also be satisfied, and consequently (assuming a smooth t dependence), the term next to the leading term of A_{Regge} can be determined.

Conclusions.—The results presented here are the following:

(i) We have lent support to the idea that the Regge term when partial-wave analyzed is a good representation of the amplitude as suggested by Schmid.[7] Most remarkable, in the reactions studied this technique has produced a new family of particles that can be naturally identified with daughters that move parallel to the leading one for a large region of t. Their presence has been demonstrated in very different reactions and, most interesting, they show up in all helicity amplitudes as required by true resonances. Ratios of couplings are in qualitative agreement with experiment and our sum rules. The possibility of saturation of sum rules with resonances only[1] is strengthened because the analysis of the Regge term shows a very small background in the imaginary part of the amplitude.

(ii) From the point of view of the sum rules, the appearance of these trajectories, necessitated by analyticity, provides contributions to the sum rules that are needed to compensate the Regge side.[2] More precisely, since a world with a single self-sustained trajectory seems impossible, this model gives a very attractive dynamical alternative. As we have shown, the saturation of the sum rules demands parallel "daughter" trajectories.

Parallel daughters of the ρ trajectory, spaced by two units of angular momentum, are also appealing because they might account as well for some of the new particles discovered in the R region.[9]

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and M. Kugler.

Note added in proof.—The symmetrization of the amplitude in formula (3) is done in order to preserve the properties of the amplitude and avoid spurious even partial waves. This parametrization contributes an oscillating term in the backward peak if this form of the amplitude is valid at all energies. However, it is also possible that our symmetrization procedure does not hold for very large s. We thank M. Kugler for discussions on this point.

[1]M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters 19, 1402 (1967).

[2]M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. (to be published).

[3]M. Bishari, H. R. Rubinstein, A. Schwimmer, and G. Veneziano, to be published.

[4]For related work see C. Schmid, Phys. Rev. Letters 20, 628 (1968); H. Harari, to be published; and P. G. O. Freund, Phys. Rev. Letters 20, 235 (1968).

[5]R. L. Sugar and J. D. Sullivan, Phys. Rev. 166, 1515 (1968).

[6]R. Swift, Phys. Rev. Letters 18, 813 (1967)

[7]C Schmid, Phys. Rev. Letters 20, 689 (1968).

[8]A detailed study of the t dependence of these sum rules is presented in Ref. 2.

[9]W. Kienzle et al., Phys. Letters 19, 438 (1965).

NEW CLASS OF DISPERSION SUM RULES FOR FORWARD SCATTERING*

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New dispersion sum rules for cross sections, coupling constants, and low-energy scattering data are derived by considering modified forward-scattering amplitudes. A set of these sum rules, which involve highly convergent integrals, may be useful for testing low-energy phase-shift analyses of $\pi\pi$ and KN scattering.

Modifications of the ordinary forward-scattering dispersion relations have been applied by many workers in the past several years. Most of these modifications have been essentially based on the Igi procedure for deriving finite-energy sum rules[1] and/or the Gilbert-Liu-Okubo technique[2-4] of working with the ordinary amplitude multiplied by a factor $e^{i\pi\beta}/q^{2\beta}$ (where β is a variable real parameter and q is the laboratory momentum). Modified dispersion relations based on the phase representation[5,6] have also been found useful in correlating high- and low-energy scattering parameters.

In two previous papers,[7,8] the present authors have made use of the knowledge of the zeros of the forward πN and KN crossing-even amplitudes to derive and apply Gilbert-type dispersion sum rules for the threshold- and infinite-energy cross sections. After completing this work, it was realized that with a slight generalization of our procedure, we could derive a very interesting set of new sum rules of which the ones derived previously from a knowledge of the zeros of the amplitudes are special cases.

In order to illustrate our procedure, we will consider the crossing-even πN and $\pi\pi$ amplitudes, but it should be evident how one should proceed in the case of other amplitudes with different crossing properties and more complicated pole and cut structures.

Let $T(\omega)\,[\equiv\frac{1}{2}(T_{\pi^+p}+T_{\pi^-p})]$ be the πN forward crossing-even amplitude normalized so that the optical theorem has the form

$$\mathrm{Im}\,T(\omega)=q\sigma(\omega), \tag{1}$$

$$q=(\omega^2-\mu^2)^{1/2}, \tag{2}$$

where μ is the pion mass, ω is the laboratory energy, and $\sigma(\omega)$ is the average of the π^+p and π^-p total cross sections. $T(\omega)$ is assumed to satisfy the once-subtracted dispersion relation

$$T(\omega)=T(\mu)+\frac{\omega^2-\mu^2}{\omega_0^{\,2}-\mu^2}\frac{2\omega_0 f^2}{(\omega_0^{\,2}-\omega^2)}+\frac{2}{\pi}(\omega^2-\mu^2)\int_0^\infty\frac{dq'}{\omega'^2-\omega^2}\sigma(\omega'), \tag{3}$$

where

$$\omega_0=\mu^2/2M, \tag{4}$$

$$f^2/4\pi\approx 0.081, \tag{5}$$

and M is the nucleon mass.

Now consider the modified amplitude

$$t(\omega)=\left[\frac{T(\omega)-T(\mu)}{(\omega^2+a^2)(\omega^2-\mu^2)^\beta}\right]e^{i\pi\beta}, \tag{6}$$