B1490 (1965); L. B. Okun', Yadern. Fiz. 1, 938 (1965) [translation: Soviet J. Nucl. Phys. 1, 670 (1965)]; J. Prentki and M. Veltman, Phys. Letters 15, 88 (1965).

[7]This cut is symmetric about the experimentally measured center of the peak.

[8]T. D. Lee, Phys. Rev. 139, B1415 (1965).

[9]C. Baltay, P. Franzini, and J. Kim, private communication.

[10]The experimental detection efficiency of our apparatus for the decay $\eta\to\pi^+\pi^-\pi^0$ was calculated using Monte Carlo techniques. As a function of the Dalitz coordinate, y, it was found to fit the approximate form $\epsilon(y) \approx 2(1-y)\%$. Note that the asymmetry as a function of x and y, $A(x,y)$, is not dependent on detection efficiency. Hence, the determination of b and c is independent of Monte Carlo calculation.

[11]C. Bemporad et al., Phys. Letters 25B, 380 (1967).

[12]We would like to thank Professor C. N. Yang for pointing out this possibility to us.

ANALYTIC STRUCTURE OF ENERGY LEVELS IN A FIELD-THEORY MODEL

Carl M. Bender*
Department of Physics, Harvard University, Cambridge, Massachusetts

and

Tai Tsun Wu
Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts

(Received 17 May 1968)

We consider a model of $\lambda\varphi^4$ field theory in which perturbation theory diverges and analytically continue the energy levels into the complex λ plane. Using WKB technique, we determine that the energy levels have an infinite sequence of branch points (where level crossing occurs) with a limit point $a\lambda=0$. Thus the origin is not an isolated singularity. The resolvent $(z-H)^{-1}$ has an infinite sequence of poles with a limit point at $\lambda=0$.

We investigate here a simple but nontrivial model of φ^4 field theory where all of the following questions can be readily answered:

(A) Is the perturbation series for the ground-state energy, which is a power series in the coupling constant λ, convergent for any $\lambda\neq 0$?

(B) If not, does the ground-state energy, considered as a function of complex λ or more generally as a function of some power α of λ, have an isolated singularity at $\lambda=0$?

(C) Is the resolvent $(z-H)^{-1}$, considered as a function of λ^α for fixed z, analytic at $\lambda=0$? If not, is the point $\lambda=0$ an isolated singularity?

The answer to all these questions is no. More precisely:

(A) The ground-state energy, which is originally defined only for positive values of λ, can be analytically continued into the complex λ plane. This analytic continuation of the ground-state energy has an infinite number of branch points, which have a limit point at the origin $\lambda=0$. Moreover, at each branch point level crossing occurs.

(B) If α is chosen to be $\frac{1}{3}$, then the resolvent has no branch cut. However, for all z, $(z-H)^{-1}$ has an infinite number of poles, which have a limit point at the origin.

The model which we consider in this paper is a φ^4 field theory of no space dimensions. The Hamiltonian is

$$H=\tfrac{1}{2}\dot{\varphi}^2+\tfrac{1}{2}m^2\varphi^2+\lambda\varphi^4. \tag{1}$$

In the interaction picture, the Fock representation for the field operator φ is

$$\varphi=(2m)^{-\frac{1}{2}}(ae^{-imt}+a^\dagger e^{imt}),$$

$$\dot{\varphi}=(\tfrac{1}{2}m)^{\frac{1}{2}}(-iae^{-imt}+ia^\dagger e^{imt}),$$

$$[a,a^\dagger]=1.$$

The perturbation series for the ground-state energy is the sum of all connected Feynman diagrams having no external legs. The Feynman rules are

$(E^2-m^2+i\epsilon)^{-1}$ for a propagator,

24λ for a vertex,

$i(2\pi)^{-1}\int_{-\infty}^{+\infty}dE$ for every loop integration.

The necessary symmetry numbers are the same as in the (3 + 1)-dimensional theory.[2]

For every diagram containing n vertices, there are $2n$ internal lines and $n+1$ loop integrations. All diagrams with n vertices add in phase and the perturbation series is finite in every order.

The number of diagrams having n vertices is at least $(n-1)!3^{-n}$ and at most $(2n-1)!!8^n$. Using Feynman integral representations, uniform upper and lower bounds may be placed on all diagrams having n vertices. Let the formal perturbation series for the ground-state energy be written as

$$E(\lambda)=\tfrac{1}{2}m-\sum_{n=1}^{\infty}\left(\frac{-\lambda}{m^3}\right)^n mE_n.$$

Then E_n are positive numbers bounded below and above by

$$A\Gamma(n)B^n<E_n<C\Gamma(n)D^n,$$

where A, B, C, and D are positive constants.[3] This implies that the above series is divergent for $\lambda\neq 0$. We thus conclude that the ground-state energy is not an analytic function of λ^α about $\lambda=0$ for any α.

The first 75 terms in the perturbation series have been calculated by computer to four-place accuracy. The result is

$$E_n\sim(1.171)(2n-1)!!(\tfrac{3}{2})^{n-1}$$

for large n.

To get more detailed information on the analytic structure of the ground-state energy, we use the coordinate representation for the Hamiltonian [Eq. (1)]:

$$\varphi=2^{-1/2}x,\quad \dot{\varphi}=-i2^{-1/2}d/dx,$$

and we let $m=1$ without loss of generality. Thus the wave function $\Psi(x)$ satisfies

$$(-d^2/dx^2+\tfrac{1}{4}x^2+\tfrac{1}{4}\lambda x^4)\Psi(x)=E(\lambda)\Psi(x) \tag{2}$$

together with the boundary conditions

$$\lim_{x\to\pm\infty}\Psi(x)=0. \tag{3}$$

Equations (2) and (3) imply that $\Psi(x)$ behaves roughly as $\exp(-\lambda^{-1/2}|x|^3)$ for large $|x|$ along the real axis. Thus Eq. (3) holds not only on the real axis but also for x in the sector $|\arg x|<\frac{1}{6}\pi$.

Until now λ has been positive. The coordinate representation allows us to continue $E(\lambda)$ into the complex plane. For complex λ, $E(\lambda)$ is defined by Eq. (2) with the boundary condition

$$\lim_{|x|\to\infty}\Psi(x)=0$$

provided that

$$|\arg(\pm x)+\tfrac{1}{6}(\arg\lambda)|<\tfrac{1}{6}\pi. \tag{3a}$$

This analytic continuation is possible provided that $\int\Psi(x)^2dx$ does not vanish. In fact, with suitable normalization

$$\int\Psi(x)^2dx=0 \tag{4}$$

is a necessary and sufficient condition for the appearance of a branch point in the λ plane for $E(\lambda)$ [except at $\lambda=0$].

The rest of this paper is devoted mainly to the determination of the approximate location of these branch points for small $|\lambda|$. For this purpose we analyze Eq. (2) using WKB techniques.[4] For large $|x|$ the WKB solution to this equation behaves like

$$\Psi(x)\sim(x^2+\lambda x^4)^{-1/4}\times\exp\{-(6\lambda)^{-1}[(1+\lambda x^2)^{3/2}-1]\}. \tag{5}$$

Figure 1 gives plots of $\mathrm{Re}\{\lambda^{-1}[(1+\lambda x^2)^{3/2}-1]\}=0$ in the complex x plane for $\arg\lambda=0$, $\frac{1}{2}\pi$, π, and $\frac{3}{2}\pi$. On these diagrams the turning points at $\pm i\lambda^{-1/2}$ are labeled by circles. Near the origin Eq. (2) becomes

$$(d^2/dx^2+E-\tfrac{1}{4}x^2)\Psi(x)=0 \tag{6}$$

which is the defining equation for parabolic cylinder functions $D_{E-\frac{1}{2}}(x)$. The correct physical solution for Eq. (6) is

$$\Psi(x)=C_1[D_{E-\frac{1}{2}}(x)\pm D_{E-\frac{1}{2}}(-x)] \tag{7}$$

for even- or odd-parity wave functions.

When the phase of λ is less than 270°, the parabolic cylinder function [Eq. (7)] must be asymptotically connected to the WKB solution [Eq. (5)].

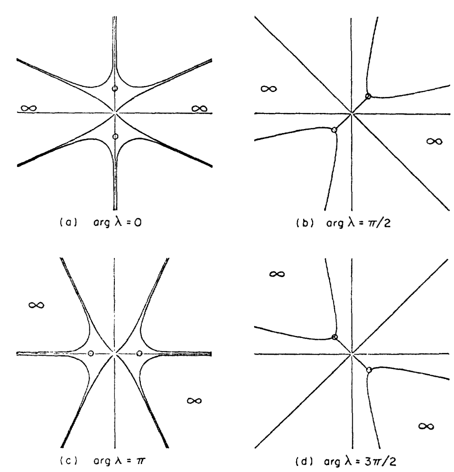

FIG. 1. Curves in the complex x plane, where $\mathrm{Re}\{\lambda^{-1}[(1+\lambda x^2)^{1/2}-1]\}=0$ for various values of $\arg\lambda$. [The circle denotes the turning point at $x=\pm i\lambda^{-1/2}$. The symbol ∞ indicates the sectors in which the boundary condition Eq. (3a) applies.]

But the condition that Eq. (7) is asymptotically Eq. (5) is a condition on E. The result is that

$$E=2n+\tfrac{1}{2},\ n=0,1,2,\cdots \text{ for even parity};$$
$$E=2n+\tfrac{3}{2},\ n=0,1,2,\cdots \text{ for odd parity}. \qquad (8)$$

However, when the phase of λ is near 270°, the above procedure is invalid. Equation (7) cannot be connected to Eq. (5) because the turning point lies in the path of the connection [see Fig. 1(a)]. Instead we use the following procedure:

(1) We define $r=xe^{\frac{1}{4}i\pi}$, $\rho=\lambda e^{-3\pi i/2}$, and $\epsilon=4iE$. In terms of these variables Eq. (2) becomes

$$[d^2/dr^2+\tfrac{1}{4}(-\epsilon+r^2-\rho r^4)]\Psi(x)=0. \qquad (2a)$$

We are assuming that $|\lambda|$ is so small that $|\rho\epsilon|\ll 1$.

(2) Also define

$$r_0=\{[1-(1-4\rho\epsilon)^{1/2}](2\rho)^{-1}\}^{1/2}\sim\epsilon^{1/2},$$
$$r_1=\{[1+(1-4\rho\epsilon)^{1/2}](2\rho)^{-1}\}^{1/2}\sim\rho^{-1/2}$$
$$=\text{position of turning point}.$$

Note that $|r_0|\ll|r_1|$.

(3) We identify four regions: (a) r near the origin, $0<|r|\ll|r_0|\ll|r_1|$; (b) $|r|\ll|r_1|$ but $|r|\gg|r_0|$; (c) r near the turning point, $|r|\lesssim|r_1|$; and (d) r at the turning point; $|r|\sim|r_1|$.

(4) In region (a) we have Eq. (6) with solution Eq. (7).

(5) In region (b) we have the following WKB solution:

$$\Psi(x)=(-\epsilon+r^2-\rho r^4)^{-1/4}\times\{C_2\exp[i\int_{r_0}^{r}2^{-1}(-\epsilon+r^2-\rho r^4)^{1/2}dr]+C_3\exp[-i\int_{r_0}^{r}2^{-1}(-\epsilon+r^2-\rho r^4)^{1/2}dr]\}.$$

Note that ϵ can no longer be neglected as it was in Eq. (5). However, we can do the integrals because the r^4 term is not important. The result is

$$\Psi(x)\sim r^{-1/2}\{C_2\exp i4^{-1}[r^2-\tfrac{1}{2}\epsilon-\epsilon\ln(2r\epsilon^{-1/2})]+C_3\exp -i4^{-1}[r^2-\tfrac{1}{2}\epsilon-\epsilon\ln(2r\epsilon^{-1/2})]\}.$$

(6) We continue the WKB solution from region (b) to (c):

$$\Psi(x)\sim(-\epsilon+r^2-\rho r^4)^{-1/4}\times\{C_2\exp i[\tfrac{1}{2}A-3^{-1}2^{1/2}\rho^{-1/4}(r_1-r)^{3/2}]+C_3\exp -i[\tfrac{1}{2}A-3^{-1}2^{1/2}\rho^{-1/4}(r_1-r)^{3/2}]\},$$

where

$$A=\int_{r_0}^{r_1}(-\epsilon+r^2-\rho r^4)^{1/2}dr,$$
$$A=3^{-1}r_1^{3}[(1+r_0^{2}r_1^{-2})E(k)-2r_0^{2}r_1^{-2}K(k)],$$

and $k^2=1-r_0^{2}r_1^{-2}$. E and K are the standard elliptic integrals.

(7) In region (d) Eq. (2) becomes an Airy equation. In this region

$$\Psi(x)=C_4y^{1/2}3^{-1/2}K_{1/3}(2y^{3/2}),$$

where $y=(18\rho^{1/2})^{-1/3}(r-\rho^{-1/2}+2^{-1}\epsilon\rho^{1/2})$.

It is extremely important that there is only one unknown constant, C_4, in this solution. Only the $K_{1/3}$ function has the correct asymptotic behavior at ∞ required by the boundary conditions.

(8) By asymptotically connecting region (a) to (b) and region (c) to (d) we have calculated the ratio C_2/C_3 in two different ways. Thus by setting these two ratios equal we have a single transcendental equation relating E and ρ. The result is

$$\frac{\Gamma(\frac{1}{4}+\frac{1}{2}E)}{\Gamma(\frac{1}{4}-\frac{1}{2}E)}=\exp\left[i3^{-1}\rho^{-1}+\frac{5\pi i}{4}-E\ln(\tfrac{1}{2}\rho)\right] \qquad (9a)$$

for even-parity wave functions;

$$\frac{\Gamma(\frac{3}{4}+\frac{1}{2}E)}{\Gamma(\frac{3}{4}-\frac{1}{2}E)}=\exp\left[i3^{-1}\rho^{-1}-\frac{5\pi i}{4}-E\ln(\tfrac{1}{2}\rho)\right] \qquad (9b)$$

for odd-parity wave functions. Equation (9) is the major result of this paper and will be used to locate the singularities in $E(\lambda)$.

We substitute the WKB solution into Eq. (4). A lengthy evaluation gives

$$(d/dE)\ln\Gamma(\tfrac{1}{2}+E)=\pi2^{-1}\cot(\tfrac{1}{4}\pi-\tfrac{1}{2}E\pi)+\ln 4/\rho \qquad (10a)$$

for even-parity wave functions;

$$(d/dE)\ln\Gamma(\tfrac{1}{2}+E)=\pi2^{-1}\cot(\tfrac{3}{4}\pi-\tfrac{1}{2}E\pi)+\ln 4/\rho \qquad (10b)$$

for odd-parity wave functions.

A straightforward but tedious analysis of Eqs. (9) and (10) gives the approximate location to first order in λ of the branch points of the analytic continuation $E(\lambda)$, for the low-lying energy levels. These branch points are precisely the branch points of Eq. (9). We summarize the results as follows:

(1) $E(\lambda)$ has branch points at

$$E=2n+\tfrac{1}{2}+\eta,$$
$$\eta=\left\{\ln(24\pi N)+\gamma-\sum_{k=1}^{2n}k^{-1}\right\}^{-1},$$

when

$$\lambda = e^{3\pi i/2}\{6\pi N + 3\pi n - \tfrac{3}{4}\pi - 3i \times[\ln(2n)! + 2^{-1}\ln\tfrac{1}{2}\pi - \ln\ln N - (2n+\tfrac{1}{2})\ln(24\pi N) - 1]\}^{-1}$$

for even parity;

$$E = 2n + \tfrac{3}{2} + \eta$$

$$\eta = \left\{\ln[24\pi(N+1)] + \gamma - \sum_{k=1}^{2n+1} k^{-1}\right\}^{-1},$$

when

$$\lambda = e^{3\pi i/2}\{6\pi(N+1) + 3\pi n + \tfrac{3}{4}\pi - 3i \times[\ln(2n+1)! + 2^{-1}\ln\tfrac{1}{2}\pi - \ln\ln N - (2n+\tfrac{3}{2})\ln[24\pi(N+1)] - 1]\}^{-1}$$

for odd parity. γ is Euler's constant and N is a positive integer.

(2) The resolvent $(z-H)^{-1}$ has poles at

$$\lambda = e^{3\pi i/2}\left\{-3i\ln\frac{\Gamma(\frac{1}{4}+\frac{1}{2}z)}{\Gamma(\frac{1}{4}-\frac{1}{2}z)} + 6\pi N + \tfrac{1}{4}\pi\right\}^{-1} \quad (11a)$$

provided that $z \neq 2N+\frac{1}{2}$; and at

$$\lambda = e^{3\pi i/2}\left\{-3i\ln\frac{\Gamma(\frac{3}{4}+\frac{1}{2}z)}{\Gamma(\frac{3}{4}-\frac{1}{2}z)} + 6\pi N - \tfrac{1}{4}\pi\right\}^{-1} \quad (11b)$$

provided that $z \neq 2N+\frac{3}{2}$. In Eq. (11) N is any sufficiently large positive integer.

That the resolvent has poles is a surprise to us and as far as we know this has not even been conjectured.[5]

$E(\lambda)$ has some interesting symmetry properties. Since Eqs. (2) and (2a) are real in their respective variables, we have

$$E(\lambda) = E^*(\lambda^*) = -E^*(e^{3\pi i}\lambda^*) = -E(e^{3\pi i}\lambda).$$

As a result of this symmetry the branch cuts are very "short."

It is tempting to conjecture that many of the qualitative features of this model are also present in a realistic field theory.

We wish to thank Professor A. M. Jaffe for many interesting discussions.

---

*Work supported by a National Science Foundation Predoctoral Fellowship.

[1]We believe that the question of convergence of perturbation series was first discussed by F. J. Dyson [Phys Rev. 85, 631 (1952)].

[2]T. T. Wu, Phys. Rev. 125, 1436 (1962).

[3]If the Hamiltonian is Wick ordered, there are fewer diagrams of order n because no internal line may have both ends connected to the same vertex. Thus the terms in the Wick-ordered perturbation series are slightly smaller than those of the non-Wick-ordered series. However this estimate holds for either perturbation series.

[4]T. T. Wu, Phys. Rev. 143, 1110 (1966). Note that WKB techniques are used here solely to locate singularities approximately and not to define the analytic continuation of $E(\lambda)$.

[5]A. M. Jaffe [thesis, Princeton University, 1965 (unpublished)] has proved that for negative z, the resolvent is analytic in the cut λ plane with a cut along the negative real axis extending from the origin to $-\infty$. This is entirely consistent with our results which maintain that nothing interesting happens until the phase of the coupling constant reaches nearly 270°.

MODEL FOR THE VIOLATION OF *CP* INVARIANCE

T. Das
Tata Institute of Fundamental Research, Bombay, India
(Received 5 June 1968)

An explicit model for CP nonconservation is constructed within the framework of the current-current form of weak interactions, which has $\Delta I = \frac{1}{2}$ for the CP-invariant part of (nonleptonic) H_w with $|\Delta S| = 1$, violates the $\Delta I = \frac{1}{2}$ rule for the CP-nonconserving part, and has no observable effects of T nonconservation in the leptonic decay modes and the electric dipole moment of the neutron.

Since the time the violation of CP invariance was first noticed[1] through the decay of the long-lived component K_L of the neutral kaon complex into $\pi^+\pi^-$, there have been considerable experimental as well as theoretical investigations about the nature and the structure of the CP-nonconserving interactions.[2] On the theoretical side, because of the elegance of the current-current theory of weak interactions with the $V-A$ structure of the currents, one naturally wishes to see