TRANSPORT PROPERTIES OF A FERMI LIQUID

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We have calculated the transport properties of a Fermi liquid by solving certain integral equations originally derived by Abrikosov and Khalatnikov. Their result for the thermal conductivity is found to be a factor of 2 too large, and the numerical agreement between theory and experiment is improved by applying this correction. The expressions for the viscosity and diffusion coefficients are also reduced from those previously accepted, but by less than 25%.

We present here a calculation of the transport properties of a Fermi liquid, based upon the derivations of Abrikosov and Khalatnikov[1] (viscosity and thermal conductivity) and of Hone[2] (diffusion). These authors obtained expressions for the transport coefficients in terms of the solutions to certain integral equations and found approximate solutions. By solving the integral equations exactly, we find that the expression for the thermal conductivity given by Abrikosov and Khalatnikov should be reduced by a factor of about 0.5; the conventional expressions for viscosity and diffusion are also reduced, but the changes are relatively small. The corrected equations are used to obtain calculated values of the transport coefficients of liquid ^{3}He, and the agreement with experiment is found to be improved. The present calculation applies to the low-temperature regime and takes no account of finite temperature phenomena, or of spin fluctuations (paramagnons).

We use the notation of Ref. 1 as far as possible. The thermal conductivity is determined by finding a quantity $q(t)=q_s(t)+q_a(t)$, where q_s is an even function of t and q_a is an odd function. Here $t=(\epsilon-\mu)/kT$, and ϵ is the quasiparticle energy, while μ is the Fermi energy; $q(t)$ is a measure of the departure of the distribution function from its equilibrium form. The equations giving $q_s(t)$ and $q_a(t)$ are

$$\int_{-\infty}^{\infty}dx\,K(t,x)[q_s(t)-q_s(x)]=-As/k, \tag{1}$$

$$\tfrac{1}{2}(\pi^2+t^2)q_a(t)-\lambda_K\int_{-\infty}^{\infty}dx\,K(t,x)q_a(x)=At, \tag{2}$$

where

$$A=\frac{8\pi^4\hbar^6}{m^{*3}kT^2}\left\{\int\frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}\right\}^{-1},$$

$$K(t,x)=\frac{(e^{-t}+1)}{(e^{-x}+1)}\frac{(x-t)}{(e^{x-t}-1)},$$

$$\lambda_K\int\frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}=\int\frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}(1+2\cos\theta).$$

Equations (1) and (2) are equivalent to (8.4) and (8.5) of Ref. 1.

We multiply Eq. (1) by dn_0/dt, where $n_0(t)=(e^t+1)^{-1}$, and integrate over t. The quantity $K(t,x)dn_0/dt$ is symmetric under interchange of t and x, while $[q_s(t)-q_s(x)]$ is antisymmetric; so the result is zero. Then s, which is defined as $-\nabla\mu/\nabla T$, is also zero (more precisely, s is of order T). By defining $Q(t)=q(t)\operatorname{sech}\frac{1}{2}t$, Eqs. (1) and (2) are transformed into

$$\tfrac{1}{2}(\pi^2+t^2)Q_s(t)-\int_{-\infty}^{\infty}dx\,g(t-x)Q_s(x)=0, \tag{3}$$

$$\tfrac{1}{2}(\pi^2+t^2)Q_a(t)-\lambda_K\int_{-\infty}^{\infty}dx\,g(t-x)Q_a(x)=At\operatorname{sech}\tfrac{1}{2}t, \tag{4}$$

where

$$g(t-x)=\tfrac{1}{2}(t-x)\operatorname{csch}[\tfrac{1}{2}(t-x)].$$

Taking the Fourier transforms of these equations, we have

$$d^2\Phi_s/dk^2+\pi^2\{2\operatorname{sech}^2(\pi k)-1\}\Phi_s=0, \tag{5}$$

$$d^2\Phi_a/dk^2+\pi^2\{2\lambda_K\operatorname{sech}^2(\pi k)-1\}\Phi_a=-4\pi^2 iA\sinh(\pi k)\operatorname{sech}^2(\pi k), \tag{6}$$

where $\Phi_s(k)$, $\Phi_a(k)$ are the transforms of Q_s, Q_a, respectively.

It is convenient to consider first the homogeneous equation

$$\Phi''+\pi^2[2\lambda\operatorname{sech}^2(\pi k)-1]\Phi=0. \tag{7}$$

This is similar to the Schrödinger equation for a particle in a sech^2-type potential well and may be solved by the methods of Landau and Lifshitz.[3] Eigenvalues $\lambda_{ns} = (n+1)(2n+1)$ (where $n = 0, 1, 2, \cdots$) are found to accompany even eigenfunctions

$$\Phi_{ns}(k) = \operatorname{sech}(\pi k) F[-n, n+\tfrac{3}{2}, 2; \operatorname{sech}^2(\pi k)],$$

while eigenvalues $\lambda_{na} = (n+1)(2n+3)$ accompany odd eigenfunctions

$$\Phi_{na}(k) = \sinh(\pi k)\operatorname{sech}^2(\pi k) F[-n, n+\tfrac{5}{2}, 2; \operatorname{sech}^2(\pi k)].$$

The hypergeometric functions occurring here are Jacobi polynomials.

The odd eigenfunctions are now used to expand the solution of Eq. (6), and give

$$\Phi_a(k) = iA \sum_{n=0}^{\infty} \frac{(4n+5)}{(n+1)(2n+3)-\lambda_\kappa} \Phi_{na}(k). \tag{8}$$

λ_κ can only lie in the range $-1 < \lambda_\kappa < 3$; so the denominator is never zero. Equation (5) used with the even eigenfunctions gives the simple result

$$\Phi_s(k) = \text{const}\,\Phi_{0s}(k).$$

The value of the constant is determined, as explained in Ref. 1, from the condition that there should be no mass flow:

$$\int dt \left(\frac{\partial \epsilon_0}{\partial p}\right)^2 \frac{\partial \tau}{\partial \epsilon} \frac{dn_0}{dt} q(t) = 0.$$

From this we readily find that $q_s(t)$ is of order T smaller than $q_a(t)$.

The thermal conductivity is given by

$$\kappa = -\frac{kT}{3} \int dt \left(\frac{\partial \epsilon_0}{\partial p}\right)^2 \frac{\partial \tau}{\partial \epsilon} \frac{dn_0}{dt} t q(t).$$

The contribution to this from $q_s(t)$ is smaller than that from $q_a(t)$ by a factor of order T^2; so $q_s(t)$ [and therefore $\Phi_s(k)$] can be ignored. Then

$$\begin{aligned}\kappa &= -\frac{kT}{3}\left[\left(\frac{\partial \epsilon_0}{\partial p}\right)^2 \frac{\partial \tau}{\partial \epsilon}\right]_{\epsilon=\mu} \int_{-\infty}^{\infty} dt \frac{dn_0(t)}{dt} t q_a(t) \\ &= -\frac{kT}{3}\left[\left(\frac{\partial \epsilon_0}{\partial p}\right)^2 \frac{\partial \tau}{\partial \epsilon}\right]_{\epsilon=\mu} \frac{i\pi}{4}\int_{-\infty}^{\infty} dk \sinh(\pi k)\operatorname{sech}^2(\pi k)\Phi_a(k), \\ &= \frac{8}{3}\frac{\pi^2\hbar^3 p_0^{\,3}}{m^{*4}T}\left\{\int \frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}(1-\cos\theta)\right\}^{-1} H(\lambda_\kappa),\end{aligned} \tag{9}$$

where

$$H(\lambda_\kappa) = \frac{(3-\lambda_\kappa)}{4}\sum_{n=0}^{\infty}\frac{(4n+5)}{(n+1)(2n+3)[(n+1)(2n+3)-\lambda_\kappa]}. \tag{10}$$

Equation (9) is the result quoted in Ref. 1, multiplied by a correction factor $H(\lambda_\kappa)$. A numerical evaluation of $H(\lambda_\kappa)$ shows that whatever assumption is made about $\omega(\theta, \varphi)$, we have $-1 < \lambda_\kappa < 3$ and

$$\tfrac{5}{12} = 0.417 < H(\lambda_\kappa) < 0.561.$$

Thus the previously accepted expression for the thermal conductivity should be multiplied by a factor of about 0.5.

An approach similar to the above can be applied to the calculation of the viscosity coefficient, and to diffusion. For viscosity we find

$$\eta = \frac{64}{45}\frac{\hbar^3 p_0^{\,5}}{m^{*4}k^2T^2}\left\{\int\frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}(1-\cos\theta)^2\sin^2\varphi\right\}^{-1} C(\lambda_\eta)$$

which is the conventional result multiplied by a correction $C(\lambda_\eta)$, given by

$$C(\lambda) = \frac{(1-\lambda)}{4}\sum_{n=0}^{\infty}\frac{(4n+3)}{(n+1)(2n+1)[(n+1)(2n+1)-\lambda]}, \tag{11}$$

Table I. Comparison of theory with experiment (after Wheatley[a]).

Pressure (atm)	$(DT^2)^{-1}$ (10^6 sec/cm^2 °K^2)		$(\kappa T)^{-1}$ (10^{-2} sec cm/erg)		$(\eta T^2)^{-1}$ (10^6 cm^2/dyn sec °K^2)	
	Expt	Theory	Expt	Theory	Expt	Theory
0.28	0.7	0.2^+	3	2	0.5	0.4
27.0	6	2	>8	7	···	1.2

[a] J. C. Wheatley, Phys. Rev. 165, 304 (1968).

where λ_η is defined by

$$\lambda_\eta \int \frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta} = \int \frac{d\Omega}{2\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}\left[i-\tfrac{3}{4}(1-\cos\theta)^2\sin^2\varphi\right].$$

Any scattering function $\omega(\theta,\varphi)$ can yield values of λ_η only in the range $-2<\lambda_\eta<1$, from which the extreme numerical values of $C(\lambda_\eta)$ are found to be 0.750 and 0.925. For diffusion, the expression obtained is

$$D=\frac{32}{3}\frac{\pi^2\hbar^6 p_0{}^2}{m^{*5}k^2T^2}(1+\tfrac{1}{4}Z_0)\left\{\int\frac{d\Omega}{4\pi}\frac{2\omega_D(\theta,\varphi)}{\cos\frac{1}{2}\theta}(1-\cos\theta)(1-\cos\varphi)\right\}^{-1}C(\lambda_D),$$

where $C(\lambda)$ is the function defined by (11), but λ_D is now defined by[4]

$$(\lambda_D-1)\int\frac{d\Omega}{4\pi}\frac{\omega(\theta,\varphi)}{\cos\frac{1}{2}\theta}=-\int\frac{d\Omega}{4\pi}\frac{\omega_D(\theta,\varphi)}{\cos\frac{1}{2}\theta}(1-\cos\theta)(1-\cos\varphi).$$

The scattering functions ω, ω_D can be expressed in terms of the antiparallel and parallel spin-scattering functions[5]:

$$2\omega(\theta,\varphi)=\omega_{\uparrow\downarrow}(\theta,\varphi)+\tfrac{1}{2}\omega_{\uparrow\uparrow}(\theta,\varphi),$$

and

$$2\omega_D(\theta,\varphi)=\omega_{\uparrow\downarrow}(\theta,\varphi).$$

λ_D is bounded by -3 and 1 (although the lower bound may not be attained), so that

$$0.750<C(\lambda_D)<0.964.$$

The correction factors $C(\lambda_\eta)$ and $C(\lambda_D)$ are only a little less than 1, and so the changes they make to the viscosity and diffusion coefficients are perhaps not very important.

Using the expressions derived above, we have corrected the theoretical values of the transport coefficients of liquid ^{3}He. In the usual way,[6] we replace $\omega_{\uparrow\downarrow}(\theta,\varphi)$ and $\omega_{\uparrow\uparrow}(\theta,\varphi)$ by their values in the forward direction ($\varphi=0$), which are then approximated by the first few terms of their Legendre function expansions in θ. The results for the parameters given by Wheatley[6] are shown in Table I. It may be seen that the viscosity and thermal conductivity are in good agreement with experiment, whereas the diffusion coefficient is too large by a factor of about 3. The correction factor $C(\lambda_D)$ for the latter case is close to 1, and has done little to help. The relatively large discrepancy in the calculated value of the diffusion coefficient may perhaps be due to the very approximate evaluation of the angular averages.

Finally, we note that the methods developed here can also be used to derive the relaxation time which is used by Abrikosov and Khalatnikov in calculating the dispersion of sound. Details of this work will be given in a separate publication.

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[2] D. Hone, Phys. Rev. 121, 669 (1961).

[3] L. D. Landau and I. M. Lifshitz, Quantum Mechanics (Pergamon Press, New York, 1965), 2nd ed., p. 69.

[4] We wish to thank Dr. V. J. Emery for pointing this out to us.

[5] D. Pines and P. Nozières, in The Theory of Quantum Liquids (W. A. Benjamin, Inc., New York, 1966), Vol. 1, p. 57.

[6] See, for example, J. C. Wheatley, in Quantum Fluids, edited by D. F. Brewer (North-Holland Publishing Company, Amsterdam, The Netherlands, 1966), p. 183.

ELECTRONIC PROPERTIES OF LIQUID WATER IN THE VACUUM ULTRAVIOLET*

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Reflectance of liquid water has been measured between 1050 and 3000 Å by two methods, one with the water in an open dish in equilibrium with its vapor and one with a CaF_2 or quartz cell. Real and imaginary parts of the index of refraction and dielectric constant, deduced from the data and Fresnel's equations, suggest an exciton transition at 8.3 eV, an interband transition at 9.6 eV, and a band gap of 9 eV.

The importance of the study of the interaction of radiation with water has been pointed out by Platzman[1]: Liquid water is perhaps the most important inorganic chemical substance and certainly the fundamental biological material. Therefore, the study of the passage of electromagnetic radiation through water is of critical importance to biology. The measurement of the optical properties of water gives information also about the interaction of charged particles with water since these two processes are related. Previous measurements of the optical properties of water have only extended down to 1700 Å because of formidable experimental difficulties in the short wavelength region of the spectrum.

We have obtained the real and imaginary parts of the complex index of refraction of water, $\tilde{n} = n + ik$, in the spectral region 1050-3000 Å.

In order to determine the optical properties of water in the vacuum ultraviolet two general methods were used. In the first the reflectance of water in equilibrium with its vapor at 1°C was measured in the spectral region 1050-3000 Å for three angles of incidence. Light from a Seya-Namioka monochromator was reflected from a gold mirror onto either the water surface or a gold reference surface at the various angles and reflected specularly onto a coronene-coated photomultiplier. The optical constants were then obtained from a solution of Fresnel's equations. This method gave reliable values of n, but the reflectances measured in this way were not accurate enough to give good values of k. These values of n are labeled "free surface" in Fig. 1.

The second method involved placing water against the plane surface of a transparent semicylinder which formed one side of a sealed cell. Light is incident normally on the curved surface, reflects from the water-semicylinder interface, and exits, again normally. A quartz semicylinder was used down to 1680 Å, where it becomes opaque, and a CaF_2 semicylinder from 1680 down to 1250 Å. The reflectance was measured as a function of angle of incidence for angles between 13° and 87°, using an angle-doubling scanning device.[2] From 1250 to 1720 Å the optical constants n and k were determined from a least-squares fit

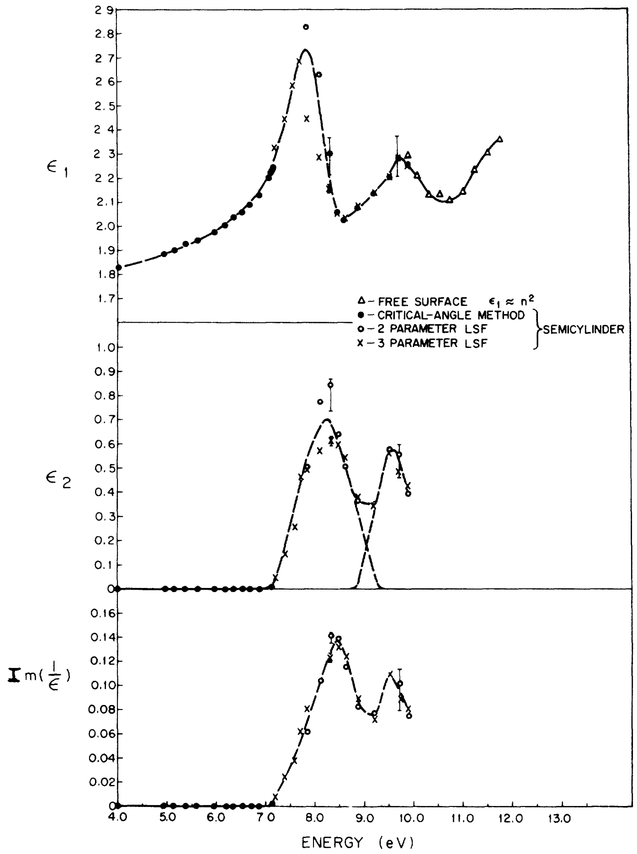

FIG. 1. Optical constants n and k of liquid water.