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[10]When some components of the symmetry remain unbroken, the mass matrix M is singular (Ref.5). In this case only its nonsingular part is to be considered. From here it is clear that only currents of broken symmetries will give the corresponding vector-meson poles. This explains why we do not have, for example, the photon pole in the matrix elements of the electromagnetic current.

FINITE-WIDTH CORRECTIONS TO THE VECTOR-MESON-DOMINANCE PREDICTION FOR $\rho \to e^+e^-$ *

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Finite-width corrections based on a generalized effective range formula for pion-pion scattering modify by a non-negligible amount the well-known relation between $\Gamma(\rho \to e^+e^-)$ and $\Gamma(\rho \to \pi\pi)$ derived on the basis of vector-meson dominance. We also present a new current-algebra prediction for the shape and magnitude of $\sigma(e^+e^- \to \pi^+\pi^-)$ and estimate the ρ-meson contribution to the Schwinger term.

It was pointed out more than six years ago[1] that the hypothesis of vector-meson dominance can be used to compute the lepton-pair decay rate of a neutral vector meson in terms of the vector-meson coupling constant appearing in strong interactions, in much the same way as the hypothesis of partially conserved axial-vector current relates the pion decay constant to the pion-nucleon coupling constant. If we call the coupling constant at the γ-ρ junction em_ρ^2/f_ρ, the complete ρ dominance of the electromagnetic form factor of $\pi^\pm$ implies

$$f_\rho = f_{\rho\pi\pi}, \tag{1}$$

or equivalently,

$$R = \frac{\Gamma(\rho \to e^+e^-)}{\Gamma(\rho \to \pi\pi)} = \frac{\alpha^2}{36}\left(\frac{m_\rho^2 - 4m_\pi^2}{m_\rho^2}\right)^{3/2}\left(\frac{m_\rho}{\Gamma_\rho}\right)^2. \tag{2}$$

In deriving the above relation, however, we have assumed that the ρ meson is essentially stable. In this note we demonstrate how Eq. (2) must be modified when we take the finite ρ width into account. We also discuss the current (field) algebra predictions on $\Gamma(\rho \to \pi\pi)$, $\sigma(e^+e^- \to \pi^+\pi^-)$, and the magnitude of the Schwinger term.

Experimentally the cleanest way to obtain the lepton-pair branching ratio R as well as the ρ-meson width is to rely on the colliding beam reaction

$$e^+e^- \to \pi^+\pi^-. \tag{3}$$

We therefore focus our attention on the electromagnetic form factor of the pion $E_\pi(s)$, which is related to the colliding-beam cross section via

$$\sigma(e^+e^- \to \pi^+\pi^-) = \frac{\pi\alpha^2}{3}\frac{(s-4m_\pi^2)^{3/2}}{s^{5/2}}|F_\pi(s)|^2. \tag{4}$$

Our starting assumption is that for a wide energy range ($s < 1$ BeV^2) the p-wave pion-pion scattering phase shift δ_1 satisfies a generalized effective-range formula of the Chew-Mandelstam type[2]:

$$(k^3/\sqrt{s})\cot\delta_1 = k^2h(s) + a + bk^2, \tag{5}$$

where

$$k = (\tfrac{1}{4}s - m_\pi^2)^{1/2},$$

$$h(s) = \frac{2}{\pi}\frac{k}{\sqrt{s}}\ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right). \tag{6}$$

Once δ_1 is given, we can write down $F_\pi(s)$ with the correct phase and singularities[3]:

$$F_\pi(s) = f(0)/f(s) \tag{7}$$

with[4]

$$f(s) = -i(k^3/\sqrt{s}) + (k^3/\sqrt{s})\cot\delta_1. \tag{8}$$

It is implied that for $0 < s < 4m_\pi^2$ we make the replacements

$$k \to i(m_\pi^2 - \tfrac{1}{4}s)^{1/2},$$

$$\ln\left(\frac{\sqrt{s}+2k}{2m_\pi}\right) \to i\cot^{-1}\left(\frac{s}{4m_\pi^2 - s}\right)^{1/2}, \tag{9}$$

so that $F_\pi(s)$ is purely real as it must be.

We now define the ρ mass m_ρ and the ρ width Γ_ρ by

$$\cot\delta_1 \big|_{s=m_\rho^2} = 0,$$

$$\frac{d\delta_1}{ds}\bigg|_{s=m_\rho^2} = \frac{1}{m_\rho \Gamma_\rho}. \tag{10}$$

With these definitions our form factor (7) can be rewritten as follows[5]:

$$F_\pi(s) = \frac{m_\rho^2 + d m_\rho \Gamma_\rho}{(m_\rho^2 - s) + \Gamma_\rho(m_\rho^2/k_\rho^3)\{k^2[h(s)-h(m_\rho^2)] + k_\rho^2 h'(m_\rho^2)(m_\rho^2 - s)\} - i m_\rho \Gamma_\rho (k/k_\rho)^3 (m_\rho/\sqrt{s})}, \tag{11}$$

where d is a constant that depends on the ρ mass:

$$d = -\frac{3}{\pi}\frac{m_\pi^2}{k_\rho^2}\ln\left(\frac{m_\rho + 2k_\rho}{2m_\pi}\right) + \frac{m_\rho}{2\pi k_\rho} - \frac{m_\pi^2 m_\rho}{\pi k_\rho^3}. \tag{12}$$

Near the ρ mass we can ignore the middle term in the denominator of (11) since it goes as $(m_\rho^2 - s)^2$, and Eq. (11) is reduced to the familiar resonance formula

$$F_\pi(s)\big|_{\text{near } s=m_\rho^2} = \frac{m_\rho^2[1 + d(\Gamma_\rho/m_\rho)]}{m_\rho^2 - s - i m_\rho \Gamma_\rho (k/k_\rho)^3 (m_\rho/\sqrt{s})}. \tag{13}$$

It is very important to note that even though our $F_\pi(s)$ is correctly normalized at $s=0$, the numerator of (13) is not just m_ρ^2. Numerically we have $d = 0.48$ for $m_\rho = 775$ MeV.

To appreciate the significance of our result let us note that the lepton-pair branching ratio R can be defined in the most unambiguous manner by the formula

$$\sigma(e^+e^- \to \pi^+\pi^-)\big|_{s=m_\rho^2} = 3\pi(2/m_\rho)^2 R. \tag{14}$$

Combining (4), (11), and (14), we get

$$R\big|_{\text{finte width}} = \frac{\alpha^2}{36}\left(\frac{m_\rho^2 - 4m_\pi^2}{m_\rho^2}\right)^{3/2}\left(\frac{m_\rho}{\Gamma_\rho} + d\right)^2, \tag{15}$$

which is to be compared with the narrow-width results (2). Although the two expressions agree in the limit $\Gamma_\rho \ll m_\rho$, for a realistic value of the ρ width they differ by as much as 15%. This is illustrated in Fig. 1. The published data of the Novosibirsk[6] and Orsay[7] colliding-beam experiments are also shown.

Recently Brown and Goble[8] have proposed that the p-wave pion-pion phase shift be obtained by matching the effective-range formula (5) to the current-algebra prediction on pion-pion scattering near threshold.[9] Their procedure leads to[10,11]

$$\Gamma_\rho = \frac{k_\rho^5}{3\pi c_\pi^2 m_\rho^2}[1 - k_\rho^4 h'(m_\rho^2)/3\pi c_\pi^2]^{-1} = 130 \text{ MeV}, \tag{16}$$

which gives rise to

$$R\big|_{\text{current algebra}} = 5.0\times 10^{-5}. \tag{17}$$

These predictions correspond to the triangular point on Fig. 1.

Using the current-algebra phase shift, we can also predict the s dependence of $|F_\pi(s)|^2$ (or equivalently the colliding-beam cross section) as shown in Fig. 2. We emphasize that the theoretical curve has no adjustable parameter once the ρ mass is given. The peak cross section obtained in the Novosibirsk experiment is in excellent agreement with our prediction, but the width observed is considerably narrower. It is worth mentioning that, when our theoretical curve is plotted versus $\sqrt{s}$, the full width at half-maximum

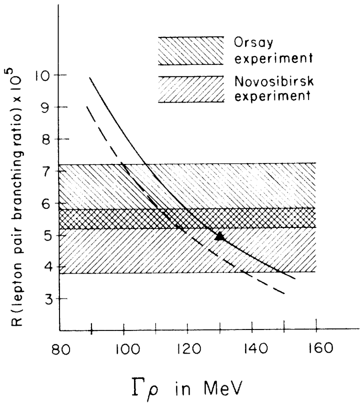

FIG. 1. The dependence of the lepton-pair branching ratio on the ρ-meson width. The full line is based on Eq. (15) which takes into account the finite ρ width. The broken line is obtained when we treat the ρ meson as a stable particle. The experimental results of the Novosibirsk and Orsay groups are indicated. The triangular point represents the current-algebra prediction.

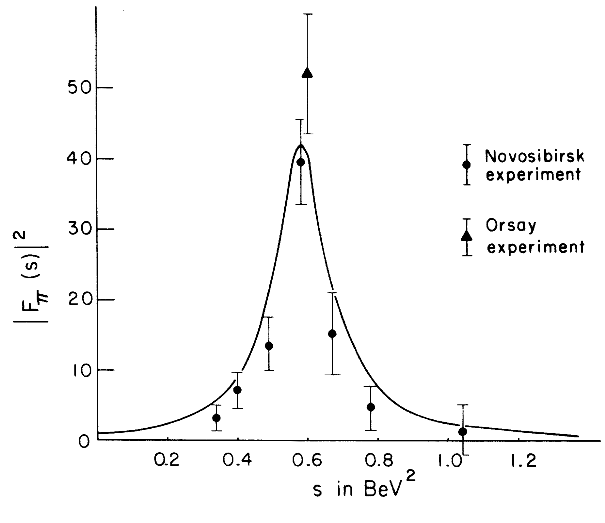

FIG. 2. Theoretically predicted $|F_\pi(s)|^2$ based on current algebra and the effective-range expansion. The theoretical curve has no adjustable parameter once the ρ mass is given ($m_\rho = 775$ MeV).

is 118 MeV even though the curve itself is based on $\Gamma_\rho = 130$ MeV in the sense of Eqs. (10) and (13). This serves to remind us once again that commonly quoted values of the ρ width often depend on the particular manners in which the experimental data are parametrized. Another interesting feature of the theoretical curve is that the actual peak of $|F_\pi(s)|^2$ is not at $m_\rho = 775$ MeV but is shifted towards the left by about 14 MeV.

The lepton-pair decay of the ρ meson is of theoretical interest also in connection with the broken SU(3) prediction[12]

$$\tfrac{1}{3} m_\rho \Gamma(\rho \to e^+e^-) = m_\varphi \Gamma(\varphi \to e^+e^-) + m_\omega \Gamma(\omega \to e^+e^-), \tag{18}$$

derived by saturating the first spectral-function sum rule of Weinberg[13] by ρ, ω, and φ in the narrow-width approximation. We can discuss finite-width corrections to the left-hand side of (18), but the cleanest way to proceed is to express the spectral-function sum rule itself in terms of the physically observable colliding-beam cross section as follows[14]:

$$\tfrac{1}{3}\int_{4m_\pi^2}^{\infty} s\sigma_{\text{tot}}(e^+e^- \to T=1 \text{ system})ds = \int_{9m_\pi^2}^{\infty} s\sigma_{\text{tot}}(e^+e^- \to T=0 \text{ system})ds. \tag{19}$$

We recall that the above cross-section integral is related to the magnitude of the Schwinger term (or the vacuum expectation value of the Schwinger term if the Schwinger term is not a c number) via[15,16]

$$c_{33} = \frac{1}{16\pi^3\alpha^2}\int_{4m_\pi^2}^{\infty} s\sigma_{\text{tot}}(e^+e^- \to T=1 \text{ system})ds, \tag{20}$$

where $c_{\alpha\beta}$ is normalized so that

$$[j_0^{\alpha}(x), j_k^{\beta}(x')]_{x_0=x_0'} = if_{\alpha\beta\gamma} j_k^{\gamma}(x)\delta^{(3)}(\vec{x}-\vec{x}') - ic_{\alpha\beta}\partial_k \delta^{(3)}(\vec{x}-\vec{x}'). \tag{21}$$

If we assume that the ρ-meson contribution saturates the integral (20), we can evaluate the Schwinger term by integrating our theoretical expression for $\sigma(e^+e^- \to \pi^+\pi^-)$ based on current algebra. The result is

$$c_{33} = 0.021 \text{ BeV}^2. \tag{22}$$

In the gauge-field algebra[17] $c_{\alpha\beta}$ is a finite constant given by

$$c_{\alpha\beta} = (m_\rho/f_\rho)^2 \delta_{\alpha\beta}. \tag{23}$$

We thus obtain[18,19]

$$f_\rho^2/4\pi = 2.3. \tag{24}$$

We may recall that in sharp contrast to the gauge-field-algebra prediction (23), the integral

(20) is expected to diverge linearly[20] in the quark-field model. The asymptotic form of the colliding-beam cross section is therefore of fundamental importance in our theoretical understanding of the hadronic current appearing in the electromagnetic interactions.

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[4]When a and b are determined to fit the mass and width of the ρ meson, $f(s)$ is seen to have a zero at $s \approx -1.2\times 10^6 m_\rho^2$. Needless to say, our formalism is expected to be valid for $|s|<1$ BeV^2; so we ignore this rather academic point.

[5]Equation (11) is essentially equivalent to the expression for the pion form factor given by W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 365 (1959), and Phys. Rev. 117, 1609 (1960). Note, however, that their ν_γ is not quite equal to our k_ρ^2.

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[8]L. S. Brown and R. L. Goble, Phys. Rev. Letters 20, 346 (1968).

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[10]The pion decay constant c_π is normalized so that it is numerically equal to 94 MeV.

[11]Equation (16) is to be regarded as a refinement on the well-known Kawarabayashi-Suzuki-Riazuddin-Fayyazuddin (KSRF) relation $c_\pi^2 = m_\rho^2/2f_{\rho\pi\pi}^2$ [K. Kawarabayashi and M. Suzuki, Phys. Rev. Letters 16, 255 (1966); Riazuddin and Fayyazuddin, Phys. Rev. 147, 1071 (1966)].

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[13]S. Weinberg, Phys. Rev. Letters 18, 507 (1967).

[14]We can obtain Eq. (19) from Weinberg's first sum rule (as applied to the third and the eighth component of the F spin)

$$\int m^{-2}\rho_1^{(3)}(m^2)dm^2 = \int m^{-2}\rho_1^{(8)}(m^2)dm^2$$

when we note that the spin-1 spectral functions are related to the colliding-beam cross sections as follows:

$$\rho_1^{(3)}(m^2) = (s^2/16\pi^3\alpha^2) \times \sigma_{\rm tot}(e^+e^- \to T=0 \text{ system})|_{s=m^2},$$

$$\rho_1^{(8)}(m^2) = (3s^2/16\pi^3\alpha^2) \times \sigma_{\rm tot}(e^+e^- \to T=0 \text{ system})|_{s=m^2}.$$

See J. J. Sakurai, Lectures on "Currents and Mesons" (The University of Chicago Press, Chicago, Ill., to be published), for details.

[15]A relation of this type was first written within the framework of qunatum electrodynamics by T. Goto and T. Imamura, Progr. Theoret. Phys. (Kyoto) 14, 396 (1955).

[16]In terms of the spectral function, Eq. (20) reads

$$c_{33} = \int m^{-2}\rho_1^{(3)}(m^2)dm^2.$$

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[18]Our coupling constant is normalized so that the current-field identity reads

$$j_\mu^{\ \alpha} = (m_\rho^{\ 2}/f_\rho)\rho_\mu^{\ \alpha} \quad (\alpha = 1,2,3).$$

[19]This value of $f_\rho^2/4\pi$ should be compared with the original KSRF value

$$f_\rho^{\ 2}/4\pi = m_\rho^{\ 2}/8\pi c_\pi^{\ 2} = 2.66$$

obtained in the narrow-width approximation.

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