relatively low powers in plasmas where collisions are rare. It is likely that this effect may occur at even lower power levels in low-pressure afterglow plasmas where both T and ν can be smaller than for the present experiment.

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SELF-TRAPPING AND INSTABILITY OF HYDROMAGNETIC WAVES ALONG THE MAGNETIC FIELD IN A COLD PLASMA

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The self-trapping and the modulational instability of nonlinear hydromagnetic waves of right-hand polarization in a cold plasma are discussed on the basis of a nonlinear dispersive equation, which enables us to directly apply the results obtained in the nonlinear optics.

The present paper deals with a long-time, asymptotic behavior of a hydromagnetic wave of small but finite amplitude propagating along an applied magnetic field in a quasineutral, cold plasma. The original equations of motion for the plasma will be reduced approximately to a dispersive nonlinear equation (the nonlinear Schrödinger equation) which takes the same form as that of the equation appearing in the self-trapping problem in nonlinear optics.[1,2] We shall thereby show marked physical effects such as the modulational instability already found in that problem.

Neglecting displacement current and assuming the charge neutrality in the first Maxwell equation gives the electron flow velocity in terms of the magnetic field and ion quantities. Then eliminating the electric field by means of the equations for the electron fluid, we can derive the following system of equations for the ion fluid and the transverse magnetic field[3]:

$$\frac{\partial n}{\partial t}+\frac{\partial(nu)}{\partial x}=0, \tag{1}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)u+n^{-1}\frac{\partial}{\partial x}\left(\tfrac{1}{2}\mathcal{B}\mathcal{B}^*\right)=0, \tag{2}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\mathcal{V}^*-n^{-1}\frac{\partial\mathcal{B}^*}{\partial x}=-iR_e^{-1}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\left(n^{-1}\frac{\partial\mathcal{B}^*}{\partial x}\right), \tag{3}$$

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\mathcal{B}^*-\frac{\partial\mathcal{V}^*}{\partial x}+\mathcal{B}^*\frac{\partial u}{\partial x}=iR_i^{-1}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\mathcal{V}^*, \tag{4}$$

$$\mathcal{V}=v+iw, \tag{5}$$

$$\mathcal{B}=B_y+iB_z \tag{6}$$

in which n is the density; u, v, and w are the x, y, and z components of the flow velocity, respectively; B_y and B_z are the components of the transverse magnetic field, while the constant applied magnetic field is oriented in the positive x direction. R_e is the electron cyclotron frequency associated with the applied field divided by a characteristic frequency Ω, and R_i is likewisely defined for the ion. All quantities are dimensionless, being normalized in terms of the applied field strength B, the density at infinity N, the Alfvén velocity $B/(4\pi\bar{m}N)^{1/2}$ ($\bar{m}$ is the ion mass plus the electron mass), and the characteristic frequency Ω.

For plane disturbances without a density change and an x component of flow velocity, $\mathcal{V}^*$ and $\mathcal{B}^*$ $\sim \exp i(kx-\omega t)$, Eqs. (3) and (4) lead to the dispersion relation[4,5]

$$(\omega/k)^2 = R_e^{-1}R_i^{-1}(R_e-\omega)(R_i+\omega). \tag{7}$$

However, superpositions of these nonlinear plane waves are not solutions; hence, for waves to transmit signals we may assume, for $\mathcal{V}^*$ and $\mathcal{B}^*$, the form $\varphi(x,t)\exp[i(kx-\omega t)]$, modulated by a slowly varying function φ. In addition, we require that the solutions include steady pulsive waves growing spatially at infinity, the condition for which may be obtained as follows: Putting $\omega = Uk$ to solve Eq. (7) for k yields that k becomes complex if $|U|$ exceeds the critical value $U_0 = (R_e+R_i)/2(R_eR_i)^{1/2}$ which is equal to the critical velocity of the solitary wave.[6] This condition implies that the phase velocity is equal to the group velocity, i.e., $\partial\omega/\partial k = \omega/k (=\pm U_0)$, and we have the critical frequency and wave number, $\omega_0 = \frac{1}{2}(R_e-R_i)$ and $|k_0| = U_0^{-1}\omega_0$, respectively. In the subsequent discussions we consider a wave of small but finite amplitude propagating with a velocity nearly equal to the critical velocity. Hence, we may assume that k is positive and that the wave has right-hand polarization.[7]

We first introduce the scale transform in terms of a slowness parameter ϵ through the equations

$$\xi = \epsilon(x-Ut), \quad \tau = \epsilon^2 t, \tag{8}$$

where U differs from U_0 by a small number $\epsilon^2\lambda$, e.g., $U = U_0+\epsilon^2\lambda$, and assume the following expansions in ϵ:

$$\begin{pmatrix} \mathcal{V}^* \\ \mathcal{B}^* \end{pmatrix} = \left(\sum_{m=1} \epsilon^m \psi_m(\xi,\tau)\right)\exp(ik_0\xi/\epsilon) \tag{9}$$

and

$$u = \epsilon^2\tilde{u}+\cdots, \quad n = 1+\epsilon^2\tilde{n}+\cdots. \tag{10}$$

Then Eqs. (1) and (2) imply that n and u are slowly varying functions. Substituting Eqs. (8)-(10) in Eqs. (3) and (4) and equating the coefficients of each power of ϵ equal to zero yields for ϵ^0,

$$W\psi_1 = 0; \tag{11}$$

for ϵ^1,

$$(W-\omega_0S_0)\psi_1' + ik_0W\psi_2 = 0; \tag{12}$$

for ϵ^2,

$$(I+k_0S_0)\partial\psi_1/\partial\tau + iU_0S_0\psi_1'' + (W-\omega_0S_0)\psi_2' + ik_0W\psi_3 + ik_0\Gamma\psi_1 = 0, \tag{13}$$

where the prime denotes the differentiation with respect to ξ, I is the unit matrix, and the matrices W, S_0, and Γ take the forms

$$W = \begin{pmatrix} -U_0 & -1+\omega_0R_e^{-1} \\ -1-\omega_0R_i^{-1} & -U_0 \end{pmatrix},$$

$$S_0 = \begin{pmatrix} 0 & -R_e^{-1} \\ R_i^{-1} & \end{pmatrix},$$

$$\Gamma = \begin{pmatrix} \tilde{u}-\lambda & \tilde{n}-k_0R_e^{-1}(U_0\tilde{n}+\tilde{u}-\lambda) \\ k_0R_i^{-1}(\tilde{u}-\lambda) & \tilde{u}-\lambda \end{pmatrix}.$$

From Eq. (11) we have the condition $|W| = 0$ which is satisfied by Eq. (7). Then ψ_1 may be represented as $\psi_1 = \varphi_1(\xi,\tau)r$ by a function φ_1 and a column vector r satisfying $Wr = 0$, say,

$$r = \begin{pmatrix} (-1+R_e^{-1}U_0k_0)/U_0 \\ 1 \end{pmatrix}.$$

Since $|W| = 0$, a compatibility condition must be satisfied so that Eq. (12) can be solved for ψ_2. Let l be the row vector satisfying $lW = 0$, say,

$$l = ((-1-R_i^{-1}U_0k_0)/U_0 \;\; 1).$$

Then multiplying Eq. (12) by l from left, we find easily that the compatibility condition $lS_0r = 0$ is satisfied by the assumed values of U_0 and k_0. In other words, this condition selects ω_0 and k_0 among ω and k satisfying the dispersion relation. We thus obtain $\psi_2 = \varphi_2(\xi,\tau)r + i\varphi_1'(\xi,\tau)s$, in which φ_2 is a function of ξ and τ and s is the column vector,

$$s = \begin{pmatrix} -R_e^{-1} \\ 0 \end{pmatrix}.$$

By means of the compatibility condition, multiplying Eq. (13) by l from the left results in

$$-i\frac{\partial\varphi_1}{\partial\tau} + \frac{k_0}{2}\{2(\tilde{u}-\lambda)-U_0\tilde{n}\}\varphi_1 + \frac{\omega_0}{2}R_e^{-1}R_i^{-1}\varphi_1'' = 0.$$

The boundary condition may be specified such that the flow in the x direction be uniform at infinity; e.g., for $x\to\infty$, n goes to unity, u vanishes, and $|\varphi_1|$ tends to a constant $|\varphi_{10}|$. Then in view of Eqs. (8)-(10) it follows immediately from Eqs. (1) and (2) that

$$\tilde{n}=(|\varphi_1|^2-|\varphi_{10}|^2)/2U_0^2,$$
$$\tilde{u}=(|\varphi_1|^2-|\varphi_{10}|^2)/2U_0.$$

Therefore we obtain the following equation for φ_1 (the nonlinear Schrödinger equation):

$$-i\frac{\partial\varphi_1}{\partial\tau}+(a|\varphi_1|^2+b)\varphi_1+c\varphi_1''=0, \tag{14}$$

where

$$a=\tfrac{1}{4}\omega_0U_0^{-2},\quad b=-\omega_0U_0^{-1}\lambda-\tfrac{1}{4}\omega_0U_0^{-2}|\varphi_{10}|^2,$$
$$c=\tfrac{1}{2}\omega_0R_e^{-1}R_i^{-1}. \tag{15}$$

This equation has the same form as that of the equation describing the self-trapping in intense light beams of slab shape,[1,2] if the index of refraction depends on the square of the electric field strength. In this case φ_1 represents the modulated transverse component of the electric field and τ and ξ are the coordinates along and normal to the direction of a beam, respectively. Hence, the results obtained in the latter case can be applied directly to our case: The solution used in Ref. 1 to represent the self-focusing, which is given by the condition $\varphi_{10}=0$, is the steady solitary wave[6]

$$\varphi_1=2(2U_0\lambda)^{1/2}\operatorname{sech}\{(2R_eR_iU_0^{-1}\lambda)^{1/2}\xi\},$$

if and only if λ is positive. Since $\tilde{n}>0$, the wave is compressional.

On the other hand, for finite values of φ_{10}, Eq. (14) admits the plane waves of the amplitude $|\varphi_{10}|$. However, these plane waves are modulationally unstable.[2,8,9] This may be shown by introducing ρ and σ through the equations $\varphi_1=\rho^{1/2}\exp[i\int\sigma d\xi/(-2c)]$ to write Eq. (14) in the form[2]

$$\frac{\partial\rho}{\partial\tau}+\frac{\partial(\rho\sigma)}{\partial\xi}=0, \tag{16}$$

$$\frac{\partial\sigma}{\partial\tau}+\sigma\frac{\partial\sigma}{\partial\xi}=2ac\frac{\partial\rho}{\partial\xi}+c^2\frac{\partial}{\partial\xi}\left(\rho^{-1/2}\frac{\partial}{\partial\xi}\rho^{-1/2}\frac{\partial\rho}{\partial\xi}\right). \tag{17}$$

Solving these equations we can see easily that the perturbation about ρ_0 and σ_0 grows for small wave number k, having the growth rate $\sim(2ac\rho_0)^{1/2}\tilde{k}$. In intense light beams the electromagnetic wave is trapped in a region of intensified polarization induced by the wave itself. In the plasma wave, particles are trapped in a region of increased attractive magnetic energy produced by the wave (note $\tilde{n}\propto|\varphi_1|^2$); then the wave is trapped by the nonlinear potential for the Schrödinger equation (14). In either case steepening of waves due to trapping is balanced by the dispersion to form a solitary wave.

It should be noted that in the present case self-steepening of waves is not caused by catching up of faster waves in the rear to slower waves in front (the characteristic crossing) but by trapping due to the nonlinear attractive potential. This may be most easily seen in the system of Eqs. (16) and (17) for ρ and σ, which does not become hyperbolic in the limit of long wavelength.[2] If the potential is repulsive in nature, namely, if ac is negative, the system admits, in this limit, the real wave velocities $\sigma\pm(-2ac\rho)^{1/2}$, leading to the characteristic crossing. In this case the system can be reduced to the Korteweg-de-Vries equation[10] and the instability does not occur.

Hence it is obvious that the instability has its origin also in the attractive nature of the nonlinear potential. For a plane wave having a uniform energy distribution, trapping does not occur; however, once the energy density is somewhere increased, particles begin to be trapped there, acting to increase the wave energy, and thus the perturbation grows. The τ evolution of the instability is not yet well understood. In the nonlinear optics, Bespalov and Talanov[8] pointed out a possibility that amplitude-phase perturbations of a plane wave bring about its decay into individual beams. Recently, Karpman[9] obtained asymptotic forms of a solution under an unstable condition. According to his results, after a sufficient τ a broad, initial hump on the amplitude of a plane wave splits into diverging solitary waves. Therefore, it seems to be possible that trains of solitary pulses resulting from perturbations propagate along the applied magnetic field.

Since the present system cannot be reduced to the Korteweg-deVries equation, it is by no means obvious that we may apply, for the evolution of interacting solitons, the results obtained from the Korteweg-deVries equation.[11] Considering the fact that in the latter case the modulational instability does not occur, we would rather deduce that the time evolution in our system has some different aspects. In this regard one is encouraged to investigate in detail the solution of the nonlinear Schrödinger equation (14).

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CAPACITANCE OBSERVATIONS OF LANDAU LEVELS IN SURFACE QUANTIZATION*

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Capacitance observations of Landau levels in a two-dimensional electron gas induced in the inversion layer on a (100) surface of p-type silicon are reported. Evidence for surface quantization and the associated lifting of the spin and valley degeneracy are presented. An observed increase in the carrier threshold with increasing magnetic field is shown to be further evidence of surface quantization.

Recently, Fowler et al.[1,2] have shown that a two-dimensional electron gas can be induced at the surface of p-type silicon using a metal-oxide semiconductor-field-effect transistor (MOSFET).[3] This two-dimensional gas is produced by quantization of the density of states shielding charge at the silicon surface, a previously anticipated result.[4-8] Fowler et al. employed the Shubnikov-de Haas effect to observe this surface quantization. Their experiments were sufficiently sensitive to observe Landau levels associated with the lifting of the electron spin and valley degeneracy predicted by Fang and Howard.[8] In this Letter we report an observation of an oscillatory capacitance due to Landau levels in a two-dimensional electron gas generated in the inversion layer on a (100) surface of p-type silicon.

In the absence of thermal or collision broadening, an electric and a magnetic field perpendicular to the surface will induce a density of states given by a series of delta functions,[9] i.e., Landau levels. Assuming neither collision broadening nor Landau-level splitting due to electron spin or valley degeneracy, the surface space charge density Q_{sc} in an inversion layer of p-type silicon is[2,9]

$$Q_{sc}=\frac{e^2H}{2\pi\hbar c}g_s g_v \times\sum_{n=0}^{\infty}\int_0^{\infty}\frac{\delta[E_{z1}+(n+\frac{1}{2})\hbar\omega_c-E]}{1+\exp[(E-E_F)/kT]}dE, \quad (1)$$

where $\delta(x)$ is the Dirac delta function of argument x, e the electronic charge, H the magnetic field perpendicular to the surface; c the velocity of light, $\hbar$ Planck's constant, g_s and g_v the spin and valley degeneracy, respectively, kT the thermal energy of the electron, E_F the Fermi energy, E_{z1} the energy of the first quantum state for $H=0$ (only the first state will be considered here), and $\omega_c=eH/m_t{}^*c$ the cyclotron frequency associated with $m_t{}^*$, the transverse effective mass of the electron. The summation is over all