particles and one of them belongs to the nonet.

An immediate consequence of the present double-pole model is the prediction that the $\pi\rho$ mass distribution resulting from an initial $K\bar{K}$ interaction should be similar to that reported in Ref. 3 for the final-state $K\bar{K}$ distribution. For example, if the K^0 or K^+ pole can be isolated in $K^-N \to A_2(\Lambda$ or $\Sigma)$, then we would expect to see a single peak in $A_2 \to \pi\rho$. As emphasized in Ref. 2, data on the $K\bar{K}$ mode are of considerable interest for a two-particle coupling model of the A_2 meson; G_1 and G_2 will vary between different reactions with different production mechanisms for the 1 and 2 components of the double pole. Thus, it might well be possible to see the $K\bar{K}$ mass distribution in some reactions as double peaked; the corresponding $\pi\rho$ distribution in this same reaction could, in general, be more complicated than that of Fig. 1(a) for the BNL $K\bar{K}$ mode. Since, as noted below Eq. (2), the fit parameters can be varied considerably, better and more data are also of interest to determine precisely how close the A_2 complex is to being an exact double-pole, which at present is technically a single point in a continuum of allowed two-pole values. The appropriate formula, in any case, is Eq. (4) of Ref. 2. For example, the assumption $\Gamma_2 \equiv 0$ can be relaxed and we find reasonable fits with $\Gamma_2 \approx 12$ MeV. This is somewhat larger than the estimate in Ref. 2 and, apparently, it should be re-emphasized, is not directly related to the width of the $K\bar{K}$ distribution.[3]

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TRAJECTORIES IN VENEZIANO'S MODEL*

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We study the trajectories in Veneziano's amplitude which give rise to true Regge behavior (without the oscillations inherent in amplitudes with very narrow resonances). It is found that the asymptotic form of the imaginary part of the trajectory is strongly constrained, but an example of a trajectory having the requisite properties is given.

Taking clues from finite-energy sum rules, Veneziano[1] has recently proposed a form for the amplitude of $\pi\pi \to \pi\omega$ which incorporates Regge behavior with linearly rising trajectories and crossing symmetry. Veneziano's discussion is formulated in the zero-resonance-width approximation, in which all the resonance poles appear on the real axis. We have studied the problem of introducing finite widths while maintaining the features of Regge behavior with linearly rising trajectories and crossing symmetry. Assuming that the trajectory $\alpha(s)$ satisfies a once-subtracted dispersion relation with a cut along the positive real axis only, we find that its imaginary part is strongly constrained. The amplitude is consistent with Regge asymptotic behavior only if

$$\lim_{s \to +\infty} \frac{\operatorname{Im}\alpha(s)}{s^{1-\mu}} = +\infty \quad \text{for all } \mu > 0,$$

but

$$\int \frac{\operatorname{Im}\alpha(s)}{s^2} ds < \infty,$$

in order that α satisfy a once-subtracted dispersion relation. The derivation of this result is indicated below, and some implications discussed. Moreover, we give an example of a trajectory which has all required features.

Veneziano's amplitude is

$$A(s,t,u)=\frac{\beta}{\pi}\left\{\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(s)-\alpha(t))}+\frac{\Gamma(1-\alpha(u))\Gamma(1-\alpha(t))}{\Gamma(2-\alpha(u)-\alpha(t))}+\frac{\Gamma(1-\alpha(u))\Gamma(1-\alpha(s))}{\Gamma(2-\alpha(u)-\alpha(s))}\right\} \quad (1)$$

with the constraint

$$\alpha(s)+\alpha(t)+\alpha(u)=2, \quad (2)$$

where

$$s+t+u=3m_\pi^{\ 2}+m_\omega^{\ 2}\equiv M^2. \quad (3)$$

Differentiating (2) with respect to s for fixed t, gives, using (3),

$$\alpha'(s)-\alpha'(M^2-s-t)=0. \quad (4)$$

Thus α' is a constant, so that

$$\alpha(s)=a+bs \quad (5)$$

with (2) implying that

$$3a+M^2b=2. \quad (6)$$

Since $\alpha(s)$ is real for negative s, both a and b are real, and this immediately leads to poles in the amplitude for large positive s, since the gamma functions in (1) are singular at nonpositive integers.

We therefore discard the constraint (2), and ask which forms of $\alpha(s)$ are consistent with Regge behavior of the amplitude A, given that

$$\alpha(s)\sim bs, \quad b>0, \quad \text{as } s\to\infty. \quad (7)$$

One sees easily that the first two terms of (1) lead to an asymptotic form

$$A_{\mathrm{I+II}}\sim\frac{\beta}{\sin\pi\alpha(t)}\frac{1}{\Gamma(\alpha(t))}[1-\cos\pi\alpha(t)-\cot\pi\alpha(s)\sin\pi\alpha(t)][\alpha(s)]^{\alpha(t)-1}, \quad (8)$$

which is the asymptotic form obtained from a Regge trajectory of negative signature if[2]

$$\cot\pi\alpha(s)\to -i \quad \text{as } s\to\infty, \quad (9)$$

i.e., if

$$\operatorname{Im}\alpha(s)\to+\infty. \quad (10)$$

We will now show that if, for some positive μ,

$$\operatorname{Im}\alpha(s)/s^{1-\mu}\to 0,$$

then the third term in (1) <u>grows</u> faster than any power of s, as s goes to infinity along the positive real axis. To demonstrate this, we invoke the following theorem, whose rather lengthy proof will be published elsewhere.

Theorem.–If (a)

$$\alpha(s)=a+bs+\frac{s}{\pi}\int_{s_0}^{\infty}\frac{\operatorname{Im}\alpha(s')}{s'(s'-s)}ds',$$

and if (b)

$$\operatorname{Im}\alpha(s)\to+\infty \text{ as } s\to+\infty,$$

(c)

$$I(s)\equiv s^{\mu-1}\operatorname{Im}\alpha(s)\to 0 \quad \text{as } s\to+\infty, \text{ for some } \mu>0,$$

(d) $I(s)$ satisfies a weak smoothness condition like

$$|I(s_1)-I(s_2)|\leq C|s_1-s_2|^{\alpha} \text{ for some } C, \quad \alpha>0 \text{ when } |s_1-s_2|\leq 1,$$

then[3] (A)

$$|\mathrm{Re}[\alpha(s)+\alpha(u)]|<C's^{1-\mu}\ln S \text{ for large } s;$$

(B) there exists a $k>0$ such that

$$-\mathrm{Re}[\alpha(s)+\alpha(u)]/\operatorname{Im}\alpha(s)\geq k \text{ for large } s.$$

Turning now to the third term of (1) for sufficiently large s, from (B)

$$-\mathrm{Re}[\alpha(s)+\alpha(u)]\to+\infty, \quad (11)$$

so that the arguments of all the Γ functions are large, and we have [recalling that $\alpha(u)$ is real and negative, and that $\alpha(s)\sim bs$ for large s with

$b>0$]

$$\left|\frac{\beta}{\pi}\frac{\Gamma(1-\alpha(s))\Gamma(1-\alpha(u))}{\Gamma(2-\alpha(s)-\alpha(u))}\right| \approx \frac{2\beta e^{-\pi\,\mathrm{Im}\,\alpha(s)}}{(2\pi)^{1/2}}\left|\frac{[-\alpha(u)]^{-\alpha(u)+\frac{1}{2}}}{[\alpha(s)-1]^{\alpha(s)-\frac{1}{2}}[1-\alpha(s)-\alpha(u)]^{\frac{3}{2}-\alpha(s)-\alpha(u)}}\right| \tag{12}$$

$$\simeq \frac{2\beta e^{-\pi\,\mathrm{Im}\,\alpha(s)}}{(2\pi)^{1/2}e^{-1}}\left|\left[1-\frac{\alpha(u)+\alpha(s)}{\alpha(s)}\right]^{-\alpha(u)+\frac{1}{2}}\right| \times\left|\left[\frac{\alpha(s)}{1-\alpha(u)-\alpha(s)}\right]^{1-\alpha(u)-\alpha(s)}\right|\left|\frac{1}{1-\alpha(s)-\alpha(u)}\right|^{\frac{1}{2}}. \tag{13}$$

By (c) and (B)

$$\left|1-\frac{\alpha(u)+\alpha(s)}{\alpha(s)}\right|^{-\alpha(u)+\frac{1}{2}} \geqslant 1, \tag{14}$$

$$\left|\left[\frac{\alpha(s)}{1-\alpha(u)-\alpha(s)}\right]^{1-\alpha(u)-\alpha(s)}\right| = \left|\frac{\alpha(s)}{1-\alpha(u)-\alpha(s)}\right|^{1-\mathrm{Re}[\alpha(u)+\alpha(s)]}\exp\left[\mathrm{Im}\,\alpha(s)\arg\frac{\alpha(s)}{1-\alpha(u)-\alpha(s)}\right] \tag{15}$$

$$\geqslant \left|\frac{\alpha(s)}{1-\alpha(u)-\alpha(s)}\right|^{1-\mathrm{Re}[\alpha(u)+\alpha(s)]} e^{-\frac{1}{2}\pi\,\mathrm{Im}\,\alpha(s)}. \tag{16}$$

But, by (A)

$$\left|\frac{\alpha(s)}{1-\alpha(s)-\alpha(u)}\right| \geqslant C''s^{\mu'}, \quad 0<\mu'<\mu, \tag{17}$$

so that

$$\left|\left[\frac{\alpha(s)}{1-\alpha(s)-\alpha(u)}\right]^{1-\alpha(u)-\alpha(s)}\right| e^{-\pi\,\mathrm{Im}\,\alpha(s)} \geqslant C''e^{-\frac{3}{2}\pi\,\mathrm{Im}\,\alpha(s)}(s^{\mu'})^{-\mathrm{Re}[\alpha(u)+\alpha(s)]} \tag{18}$$

$$\geqslant C''e^{-\frac{3}{2}\pi\,\mathrm{Im}\,\alpha(s)}(s^{\mu'})^{k\,\mathrm{Im}\,\alpha(s)} \tag{19}$$

$$\geqslant C''(e^{-\frac{3}{2}\pi}s^{\mu' k})^{\mathrm{Im}\,\alpha(s)} \tag{20}$$

which becomes infinite faster than any power of s since $\mathrm{Im}\,\alpha(s)\to+\infty$. This establishes the result.

It is possible to give an example in which $\mathrm{Im}\,\alpha$ does grow almost as fast as s, and where the third term is well behaved. Suppose

$$\mathrm{Im}\,\alpha(s) = s/(\ln s)^{\nu}, \quad \nu>1, \quad s\geqslant s_0>1. \tag{21}$$

Then one shows explicitly that

$$-\mathrm{Re}[\alpha(s)+\alpha(u)] = Ks/(\ln s)^{\nu+1} + O(s/(\ln s)^{\nu+2}), \tag{22}$$

where

$$K = \int_0^{\infty}\frac{\ln\lambda\, d\lambda}{(\lambda-1)(\lambda+1)} > 0. \tag{23}$$

Then

$$\left|\left[1-\frac{\alpha(u)+\alpha(s)}{\alpha(s)}\right]^{-\alpha(u)}\right| \approx \exp[2\nu Ks/(\ln s)^{\nu+1}]\times\text{lower order terms}, \tag{24}$$

$$\left|\left[\frac{\alpha(s)}{1-\alpha(s)-\alpha(u)}\right]^{1-\alpha(u)-\alpha(s)}\right| \approx \exp[2\nu^2 Ks\ln\ln s/(\ln s)^{\nu+1}]\times\text{lower order terms}, \tag{25}$$

whereas

$$e^{-\pi \operatorname{Im}\alpha(s)} \approx \exp[-\pi s/(\ln s)^{\nu}], \tag{26}$$

so that the product of the three goes to zero faster than any power of s. Notice that in this case

$$\alpha(s) \approx bs + \frac{s}{\pi}\frac{1}{\nu-1}\frac{1}{(\ln s)^{\nu-1}} + \text{lower order terms.} \tag{27}$$

Therefore,

$$[\alpha(s)]^{\alpha(t)-1} \approx (bs)^{\alpha(t)-1}\left[1 + \frac{\alpha(t)-1}{\pi b(\nu-1)(\ln s)^{\nu-1}} + \text{lower order terms}\right], \tag{28}$$

which means that we have not only poles in the l plane, but cuts as well, whose end point is $l = \alpha(s)$. This possibility was anticipated by Veneziano.[1]

Because the constraint (2) is not satisfied in these models, there will be poles in the amplitude at the positive even integers. This indicates the existence of a trajectory displaced from the leading trajectory by one unit, but with the same signature. There are, however, no multiplicative fixed poles at the wrong-signature nonsense points.[4]

A model containing the multiplicative wrong-signature nonsense poles, without the poles at $\alpha(s) = 2n$, has been proposed by Virasoro.[5] Arguments similar to those presented here show that there are no forms of the trajectory which are consistent with Virasoro's amplitude.

Most authors[6] who discuss the implications of linearly rising trajectories conclude that the widths go to zero rapidly as s increases because of centrifugal-barrier arguments. In our example this is not the case. If we can relate the width to $\operatorname{Im}\alpha$ in the usual way,

$$m\Gamma \sim \operatorname{Im}\alpha(s)/\alpha'(s), \tag{29}$$

we find

$$\Gamma \sim s^{\frac{1}{2}}/(\ln s)^{\nu}. \tag{30}$$

The physical significance of this relation is not clear. Since the spacing between successive resonances becomes much smaller than the width, it becomes impossible to pick out the effects of a single resonance. Probably then for large energy, the resonance region washes out into some smooth background.

A proof of the theorem and a more thorough discussion of the implications of the example presented here will be published elsewhere.

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[3](B) should actually be stated as follows: There is an infinite sequence $s_n \to \infty$ for which

$$-\operatorname{Re}\{\alpha(s_n) + \alpha[u(s_n)]\}/\operatorname{Im}\alpha(s_n) \geq k > 0.$$

But this is sufficient for the subsequent argument.

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