lar distribution best shows the interference between Λ(1405) and Λ(1520). Figure 2(b) shows the A_2/A_0 ratio that arises primarily from the S and D states and their interference, since the P amplitudes are small compared with S and D. The curves correspond to the two choices of relative sign of Λ(1405) with respect to Λ(1520). It is clear from the data that beyond resonance the two amplitudes must be nearly in phase, as is required if both resonances are SU(3) singlets.

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[1]R. Tripp et al., Nucl. Phys. B3, 10 (1967). An updated version of these SU(3) comparisons of baryon resonances will appear in the Proceedings of the Fourteenth International Conference on High Energy Physics, Vienna, Austria, 1968, to be published.

[2]The method of employing relative signs for SU(3) classification was first used by A. Kernan and W. Smart, Phys. Rev. Letters 17, 832 (1966).

[3]M. Watson, M. Ferro-Luzzi, and R. Tripp, Phys. Rev. 131, 2248 (1963).

[4]There are no indications that higher partial waves appear in detectable amounts at the low energies considered here. We have altered slightly the analysis method of Ref. 3 by employing the appropriate c.m. momentum for each channel in the decay widths of the resonant amplitudes, using a fixed radius of interaction of 1 F in the expression for the centrifugal barrier.

[5]Σ^+ polarizations from this experiment have been published in R. Bangerter et al., Phys. Rev. Letters 17, 495 (1966).

[6]We reverse the convention used in Ref. 3 to make it more appropriate for discussion of unitary symmetry.

[7]J. J. de Swart, Rev. Mod. Phys. 35, 916 (1963).

[8]From the known branching fractions of Λ(1520) and Λ(1690) (a presumed member of the $\frac{3}{2}^-$ octet) there is reasonably good evidence for some singlet-octet mixing between these two states, as noted by Ref. 1 and by G. B. Yodh, Phys. Rev. Letters 18, 810 (1967). In order to alter the sign of the resonant Λ(1520) amplitude the mixing angle would have to exceed about 45 deg. The estimate for this angle is about 16 deg. For Λ(1405) there is evidence for much stronger SU(3) symmetry breaking, as discussed later in this Letter. The conclusion of our experiment concerning Λ(1405) is that the symmetry breaking is not so severe as to alter the sign of the resonant amplitude.

[9]J. Kadyk, Y. Oren, G. Goldhaber, S. Goldhaber, and G. Trilling [Phys. Rev. Letters 17, 599 (1966)] have investigated the $\Lambda\pi$ channel in the reaction $K_2^0 p \rightarrow \Lambda\pi^+$ at low energies and find that the angular distributions require a substantial P_{13} amplitude whose magnitude is consistent with the tail of Σ(1385). A similar interpretation based on their data and those of Ref. 3 has been put forward by J. K. Kim, Phys. Rev. Letters 19, 1074 (1967). We may also note that there are at present no other known P_{13} resonances which could contribute a substantial amplitude in the vicinity of 1520 Mev.

[10]The ratio $\chi^2/\langle\chi^2\rangle$ (where $\langle\chi^2\rangle$ is the expected chi squared) for the constant-scattering-length parametrization is 1.23, indicating either that detectable differences with this constraining parametrization are beginning to appear, or that the data in their preliminary form have some additional biases. The resonant parametrizations have yielded ratios of 1.38 for P_{13} parametrized as a resonance and 1.85 for S_{01} parametrized as a resonance.

[11]Particle Data Group, University of California Radiation Laboratory Report No. UCRL-8030 Revised, 1968 (unpublished).

[12]C. Weil, Phys. Rev. 161, 1682 (1968). For this Λ(1405) ratio which should be 1 in exact SU(3) Weil obtains 11.1 by use of the S_{01} scattering lengths of J. K. Kim [Phys. Rev. Letters 14, 29 (1965)]. A more recent analysis by J. K. Kim and F. von Hippel (to be published) yields 6.8 ± 1.0 for this ratio.

VENEZIANO FORMULA WITH TRAJECTORIES SPACED BY TWO UNITS*

Stanley Mandelstam
Department of Physics, University of California, Berkeley, California
(Received 30 October 1968)

By representing a scattering amplitude as a sum of terms of the Veneziano type, we can cancel alternate trajectories and remain with trajectories spaced by two units of angular momentum. This result is obtained without imposing a supplementary condition and without introducing poles in the Regge residues at nonsense wrong-signature integers.

Veneziano[1] has proposed a formula for a scattering amplitude which has Regge asymptotic behavior in all channels. The Regge trajectories in such an amplitude are normally spaced at unit distance from one another in α, but Veneziano has proposed a supplementary condition[2] on the trajectory parameters which would remove alternate trajectories. A somewhat different formula for an amplitude with Regge asymptotic behavior in all channels has been proposed by Virasoro.[3] Virasoro's amplitude has trajectories spaced by two units, even though he does not impose a supplementary condition. On the other hand, the Regge residues have poles at the negative wrong-signature integers. Such poles would not be allowed by unitarity, since an amplitude cannot have cuts in the j plane in the narrow-resonance approximation.

It is not obvious that one should remove the alternate trajectories, since an amplitude with trajectories spaced by one unit of angular momentum has no unphysical features. However, one may want to begin constructing a scattering amplitude with the minimum number of trajectories; one would add further trajectories if they are necessitated by the imposition of additional physical requirements. It is therefore of interest to determine whether or not one requires the odd trajectories.

In this note we wish to show that one can eliminate the alternate trajectories from the Veneziano model by representing the amplitude as a sum of terms, each of which is similar to the amplitude of the original model. If no supplementary condition is imposed on the trajectory parameters, the sum will contain an infinite number of terms but will converge for all values of s and t. By imposing a generalization of the Veneziano supplementary condition, one can ensure that the sum has a finite number of terms and, by imposing the Veneziano supplementary condition itself, one restricts the sum to a single term. Since the formula with an infinite number of terms has all the desirable properties of the original Veneziano formula, we have no clear reason for imposing the Veneziano supplementary condition, even if we wish to eliminate the alternate trajectories. We shall observe that the formula with an infinite number of terms has extra terms in its asymptotic expansion along the real axis, though it has a normal Regge asymptotic behavior in the complex plane. This may lead one to keep in mind the possibility of applying the generalized supplementary condition but, in any case, we see no reason for applying the original Veneziano supplementary condition.

The Veneziano formula for spinless particles is

$$A(s,t)=\mathrm{B}(-as-b,\,-at-b),\tag{1}$$

where $as+b$ are the Regge trajectory functions, assumed linear, and B is the beta function. By taking the sum of three such terms with the pair of variables st replaced by su and tu, one can satisfy crossing in all three variables s, t, and u, but the simple formula (1) will be sufficient for our purposes. If the s and t channels are not identical, the two intercepts b may be different. The two slopes a must be the same, however, otherwise Eq. (1) would lead to an increasing exponential behavior as s or t approached infinity in the physical region with the scattering angle fixed and within a certain range.

We now generalize the Veneziano formula by taking a sum of terms, each of the original form:

$$A(s,t)=\sum_r a_r \mathrm{B}(-as-b+r,\,-at-b+r).\tag{2}$$

If the summation is taken over positive integers (including zero), the Regge trajectories of all terms will coincide, so that no new trajectories will be introduced. In fact, we shall now show that it is possible to cancel alternate trajectories by suitably adjusting the constants a_r.

We shall examine the asymptotic expansion of the series (2) by using the integral representation of the B function. By doing so we shall also be able to obtain our modified Veneziano formula (without alternate trajectories) in closed form. The integral representation for the beta function is as follows:

$$\mathrm{B}(-as-b,\,-at-b)=\int_0^1 dx\, x^{-at-b-1}(1-x)^{-as-b-1}.\tag{3}$$

The expression on the right of Eq. (2) will therefore have the integral representation

$$A(s,t)=\int_0^1 dx\, x^{-at-b-1}(1-x)^{-as-b-1}\{1+\sum_{r=1}^{\infty} a_r x^r(1-x)^r\}.\tag{4}$$

To find the asymptotic behavior in s of the integral on the right of (4), it is convenient to make the transformation

$$x=1-e^{-y}.$$

Furthermore, let us replace the variable s by

$$w=\tfrac{1}{2}z_t(t-4\mu^2)$$
$$=s+\tfrac{1}{2}(t-4\mu^2).\tag{5}$$

Equation (4) then becomes

$$A(s,t)=\int_0^\infty dy\, e^{awy}(2\sinh\tfrac{1}{2}y)^{-at-b-1}e^{\delta y}\{1+\sum_{r=1}^{\infty}a_r e^{-ry}(1-e^{-y})^r\}, \tag{6}$$

where

$$\delta=\tfrac{1}{2}(4a\mu^2+3b+1). \tag{7}$$

We shall begin by examining the behavior of A as s or w approaches $-\infty$, since we can then use Eq. (6) without deforming the path of integration. When the variable w is large and negative, the important part of the range of integration on the right of (6) will be $|y|\leqslant 1$. The factor $e^{\delta y}$ as well as the expression within curly brackets will then be equal to unity, but the factor $(2\sinh\frac{1}{2}y)^{-at-b-1}$ will behave like $y^{-at-b-1}$. We therefore rewrite (6) as

$$A(s,t)=\int_0^\infty dy\, y^{-at-b-1}e^{awy}\{(2\sinh\tfrac{1}{2}y)/y\}^{-at-b-1}e^{\delta y}\{1+\sum_{r=1}^{\infty}a_r e^{-ry}(1-e^{-y})^r\}. \tag{8}$$

The asymptotic behavior in w of the integral on the right of (8) can be found by using the formula

$$\int_0^\infty dy\, y^{\nu-1}e^{awy}=(-aw)^{-\nu}\Gamma(\nu)\quad(\mathrm{Re}\,aw<0). \tag{9}$$

Since the last three factors of the integrand on the right of (8) are unity when $|y|\ll 1$, we can immediately write down the result:

$$A(s,t)\sim(-aw)^{at+b}\Gamma(-at-b)w\to-\infty. \tag{10}$$

By expanding the last three factors in (8) as power series in y and using (9), we can obtain an asymptotic expansion for A in the form

$$A(s,t)\sim(-aw)^{at+b}\{\sum_{r=0}^{\infty}b_r w^{-r}\}. \tag{11}$$

If the Regge trajectories of the amplitude A are to be spaced by two units instead of by one, the series in (11) should contain only terms with even r. This will be the case if the product of the last three factors in (8) can be expanded in a power series in y with only even powers. The factor $\{(2\sinh\frac{1}{2}y)/y\}^{-at-b-1}$ already has this property, and it is easy to choose constants a_r in such a way that the product of the last two factors is an even function of y. We simply impose the condition

$$\{1+\sum_{r=1}^{\infty}a_r e^{-ry}(1-e^{-y})^r\}=\{1-e^{-y}(1-e^{-y})\}^{\delta}. \tag{12}$$

The product of the last two factors of (8) is then

$$\{e^{y}+e^{-y}-1\}^{\delta}, \tag{13}$$

which is an even function of y. Having thus defined our power series, we can write Eq. (4) as

$$A(s,t)=\int_0^1 dx\, x^{-at-b-1}\times(1-x)^{-as-b-1}\{1-x(1-x)\}^{\delta}, \tag{14}$$

where the constant δ is defined by Eq. (7). Equation (14) provides us with the required formula for an amplitude having trajectories spaced by two units. The Veneziano supplementary condition states that $\delta=0$ and, if it is fulfilled, our formula reduces to the original Veneziano formula.

By expanding the last factor of (14) as a power series in $x(1-x)$ and using the integral representation for the beta function, we can write (14) in the form (2):

$$A(s,t)=\sum_{r=0}^{\infty}(-1)^r\frac{\delta(\delta-1)\cdots(\delta-r+1)}{r!}\mathrm{B}(-as-b+r,\,-at-b+r). \tag{15}$$

If we wish we may write Eq. (15) as

$$A(s,t)=\mathrm{B}(-as-b,\,-at-b)\,{}_3F_2(-as-b,\,-at-b,\,-\delta;\,-\tfrac{1}{2}(as+at+2b),\,-\tfrac{1}{2}(as+at+2b-1);\,\tfrac{1}{4}). \tag{16}$$

Equation (16) will probably not be of much use, however, since the properties of the generalized hypergeometric function have not been studied extensively.

In general we would not expect the scattering amplitude to be represented by a single expression (14) or (15), since a sum of such terms with $as+b+1$ and $at+b+1$ replaced by $as+b+1-2r$ and $at+b+1-2r$ would be equally satisfactory. The coefficients of these terms might be determined from a multichannel bootstrap in which the resonances of the scattering amplitude are also the external particles. We do not wish to examine such questions in this Letter; we merely mention them to emphasize that the complete series on the right of (15) should not be taken too seriously. By taking the first $2r-1$ terms of (15) we can ensure that the first r odd trajectories are absent, and that should be sufficient for practical purposes. One could alternatively work with the closed form (14), but to do so would probably be considerably more complicated than to use a few terms of (15). On the other hand, the closed form (14) is useful for investigating general properties of the scattering amplitude and, in particular, for investigating the asymptotic behavior.

Our next step is in fact to re-examine the asymptotic behavior of (14) as s approaches infinity. We had previously restricted ourselves to the limit $s\rightarrow-\infty$ and, further, we had only examined the asymptotic behavior of the individual terms obtained by expanding the last factor, which may be different from the asymptotic behavior of the complete amplitude. We shall not reproduce the detailed calculation, which is fairly simple and straightforward and is similar to the calculation leading to Eq. (10). Depending on the manner in which s approaches infinity, and depending on the variable which is held constant during the process, we may have to deform the path of integration in (14). We obtain contributions to the asymptotic behavior from the regions $|x|\ll 1$ and $|x|\gg 1$, and also from the regions $x\approx e^{\pm i\pi/3}$, where the expression in the curly brackets of (14) vanishes. When s approaches infinity in any direction other than along the positive real axis, the results are exactly the same as in the original Veneziano model, and we need not repeat them here. When s approaches infinity along the positive real axis, we obtain two contributions:

$$A(s,t)\approx A^{(1)}(s,t)+A^{(2)}(s,t), \qquad (17)$$

where $A^{(1)}$ is the contribution from $|x|\ll 1$ or $|x|\gg 1$ in the integral (14), and $A^{(2)}$ is the contribution from $x\approx e^{\pm i\pi/3}$. We find that

$$A^{(1)}(s,t)\approx\{(as+b+1)^{at+b}\Gamma(-at-b)\sin\pi(as+b+1+at+b+1)\}/\sin\pi(as+b+1),\quad s\rightarrow\infty,\ t\ \text{const}; \qquad (18a)$$

$$A^{(1)}(s,t)\approx\{(as+b+1)^{au+b}\Gamma(-au-b)\}\sin\pi(au+b+1)/\sin\pi(as+b+1),\quad s\rightarrow\infty,\ u\ \text{const}; \qquad (18b)$$

$$A^{(2)}(s,t)\sim\{-2.3^{\frac{1}{2}\delta}(as+b+1)^{-\delta-1}\Gamma(\delta+1)\sin\pi\delta\cos\tfrac{1}{3}\pi[2(as+b+1)+(at+b+1)-\tfrac{7}{2}\delta-2]\}/\sin\pi(as+b+1),\quad s\rightarrow\infty. \qquad (18c)$$

The formulas (18a) and (18b) are exactly the same as the formulas we would have obtained from the ordinary Veneziano model with the supplementary condition. In fact, if we have a formula with an st term and an su term, so that we can replace u by t in (18b) and add it to (18a), we obtain the result

$$A^{(1)}(s,t)=(as+b+1)^{at+b}\Gamma(-at-b)\sin\pi[\tfrac{1}{2}(as+b+1)+at+b+1]\}/\sin\tfrac{1}{2}\pi(as+b+1),\quad s\rightarrow\infty,\quad t\ \text{const}. \qquad (19)$$

This resembles a Regge asymptotic formula, except that the continuous imaginary part is replaced by a sum of delta functions at $as+b+1=2r$.

The extra term (18c) decreases exponentially when s goes to infinity in a complex direction, but oscillates when s goes to infinity along the real axis. If s goes to infinity at fixed t, the period of these oscillations is $3/a$, instead of $1/a$ as it was with (18b).

If the constant δ, defined by (7), is a positive integer, the series (15) will terminate. The singularity at $x=e^{\pm i\pi/3}$ in the integrand of (14) will be absent, and the term (18c) in the asymptotic formula becomes zero. We must emphasize that there is no reason of principle why a term (18c) should be excluded. It does not invalidate the Regge asymptotic formula when s approaches infinity in a complex direction, and the Veneziano model does not in any case satisfy the Regge asymptotic formula on the real axis, except in an average sense. It may possibly turn out that a term (18c) is undesirable from the point of view of satisfying bootstrap restrictions. If so, one would impose the supplementary condition

$$4a\mu^2 + 3b = -1 + 2n, \qquad (20)$$

where n is a positive integer. The special case $n = 0$ corresponds to the Veneziano condition. The large period of oscillation of the term (18c) may mean that the finite-energy sum rules are badly violated if the cutoff is low. This is perhaps the reason why results obtained from the Veneziano supplementary condition agree with those found by using finite-energy sum rules.

We should like to acknowledge helpful discussions with G. F. Chew, J. Shapiro, and J. Yellin.

Note added in proof.—It has been found by Shapiro (private communication) that the formula (14) can give negative residues to certain of the lower resonances. This re-emphasizes the fact that the complete amplitude must consist of a sum of such terms with $as + b + 1$ and $at + b + 1$ replaced by $as + b + 1 - 2r$ and $at + b + 1 - 2r$.

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[1]G. Veneziano, Nuovo Cimento 57A, 190 (1968).

[2]See Veneziano, Ref. 1, Eq. (7).

[3]M. A. Virasoro, to be published.

ERRATA

ENHANCED SUPERCONDUCTIVITY IN LAYERED METALLIC FILMS. Myron Strongin, O. F. Kammerer, J. E. Crow, R. D. Parks, D. H. Douglass, Jr., and M. A. Jensen [Phys. Rev. Letters 21, 1320 (1968)].

It should be noted that a correlation between hardness and T_c has been previously made by Matthias.[1] Matthias points out that hardness is at a minimum near $4Nb_3Al{:}1Nb_3Ge$, the composition where T_c reaches 21°K. Also, for the Zr-Os system the maximum T_c occurs where the ductility is a maximum.

[1]B. T. Matthias, Phys. Letters 25A, 226 (1967).

K^+d PARTIAL CROSS SECTIONS AROUND 1 BeV/c. Allen A. Hirata, Charles G. Wohl, Gerson Goldhaber, and George H. Trilling [Phys. Rev. Letters 21, 1485 (1968)].

In Eq. (6), the fourth reaction should read $K^+n \to K^0\pi^+n$; the seventh should read $K^+n \to K^+\pi^-p$. In Eq. (8), the first reaction on the right-hand side should read $K^+n \to K^0\pi^+n$.