found to be important. The assignment of shell numbers had to be changed for some lines. The results will be published elsewhere.

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[6]The displacements of neighboring lines from their calculated values must be considered in making this comparison.

TRANSVERSE HALL EFFECT IN THE ELECTRIC QUANTUM LIMIT

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When a current density $\vec{J}$ and a magnetic induction $\vec{B}$ are in the plane of a layer thin enough to quantize the motion perpendicular to the plane of the layer, the Hall voltage arises from deformation of the quantized states. If the layer is asymmetric, the Hall voltage will in general contain a term in B^2, as well as a term in $\vec{J}\times\vec{B}$. Numerical results are given for n-type inversion layers on (100) silicon surfaces.

When a thin layer carries a current in the plane of the layer in the presence of a magnetic field perpendicular to the current and parallel to the plane of the layer, the Hall voltage appears across the small dimension of the layer. We call this the transverse Hall effect; it is Sondheimer's[1] case C1, which was treated by MacDonald and Sarginson[2] without considering quantization of the states.

We consider for the first time the effect of quantization on the transverse Hall effect. The Hall voltage in this case arises from a deformation of the quantized states in the presence of a magnetic field. We limit the analysis to the electric quantum limit,[3] in which all the carriers are in the lowest electronic subband associated with the direction perpendicular to the plane of the layer, but move as free carriers in the plane of the layer. A perturbation treatment is given for a general case, and more detailed results, including a self-consistent field treatment, are given for inversion layers on silicon surfaces.

Let electrons of charge $-e$ move in a layer whose states are quantized in the z direction by a potential $V(z)$, and let their energies and wave functions in the absence of a magnetic field be given by the solutions of the effective-mass equation

$$[p_x^2/2m_1 + p_y^2/2m_2 + p_z^2/2m_3 + V(z)]\psi = E\psi, \quad (1)$$

$$E = E_n'' + \hbar^2 k_x^2/2m_1 + \hbar^2 k_y^2/2m_2, \quad (2)$$

$$\psi = \zeta_n(z)\exp(ik_x x + ik_y y), \quad n = 0, 1, 2, \cdots, \quad (3)$$

where n labels the subbands. In the presence of a magnetic induction B_y in the y direction, the Hamiltonian has the additional term

$$H_1 = b_y z p_x/m_1 + b_y^2 z^2/2m_1, \quad (4)$$

where we use the gauge $\vec{A} = (B_y z, 0, 0)$, mks units, and $b_y = eB_y$.

The perturbed energy to second order in b_y is

$$E = E_n'' + \frac{\hbar^1 k_y^2}{2m_2} + \frac{(\hbar k_x + b_y z_{nn})^2}{2m_1} + \frac{b_y^2(z^2{}_{nn} - z_{nn}^2)}{2m_1} + \frac{\hbar^2 k_x^2 b_y^2}{m_1^2} \sum_i{}' \frac{|z_{in}|^2}{E_n'' - E_i''}, \quad (5)$$

where $z^p{}_{ij} = (\zeta_i | z^p | \zeta_j)$, the ζ_i are the normalized functions of Eq. (3), and the primed sum excludes vanishing energy denominators. Except for the last term, which gives a change in the effective mass in the x direction, this expression agrees with the first-order perturbation theory result in Appendix

A of Stern and Howard.[3]

The wave function is also changed by the magnetic field. The average value of z in the perturbed state with quantum number n and wave vector k_x in the x direction is

$$z_{nn}+\frac{2\hbar k_x b_y}{m_1}\sum_i{}' \frac{|z_{ni}|^2}{E_{ni}}+\frac{b_y^2}{m_1}\sum_i{}' \frac{z_{ni}z^2_{in}}{E_{ni}}+\frac{3\hbar^2 k_x^2 b_y^2}{m_1^2}\left(\sum_{i,j}{}' \frac{z_{ni}z_{ij}z_{jn}}{E_{ni}E_{nj}}-\sum_i{}' \frac{z_{nn}|z_{in}|^2}{E_{ni}^2}\right), \tag{6}$$

where $E_{ij}=E_i''-E_j''$. Note that if the potential $V(z)$ in Eq. (1) has inversion symmetry, so that the states have either even or odd parity, all the terms in b_y^2 vanish, and only the linear term remains.

In the electric quantum limit, only the lowest electronic subband, with $n=0$, is occupied. We can obtain an approximate result for this case by replacing the energy denominators by a constant D, and by using the closure relation[4] to simplify the numerators. Then (6) becomes

$$z_{00}-2\hbar k_x b_y(z^2{}_{00}-z_{00}{}^2)/m_1 D-(b_y{}^2/m_1 D)(z^3{}_{00}-z^2{}_{00}z_{00})$$
$$(3\hbar^2 k_x{}^2 b_y{}^2/m_1{}^2 D^2)(z^3{}_{00}-3z^2{}_{00}z_{00}+2z_{00}{}^3). \tag{7}$$

The Hall voltage is directly related to the deformation of the charge distribution. A change Δz in the average value of z for the carriers produces a voltage change across the layer given by

$$V_H=Nq\Delta z/\epsilon, \tag{8}$$

where ϵ is the permittivity of the layer; N is the number of carriers per unit area in the deformed states; and q is their charge, which is $-e$ in our case. The quantum-mechanical Hall voltage obtained from perturbation theory for the electric quantum limit is thus given by (8), where Δz is the change in the average value of z from the value z_{00} in the absence of a magnetic field to the value given by (6) or (7). To average over the occupied values of the wave vector $\vec{k}$, we use the effective-mass change indicated by (5), and shift the Fermi ellipse along the k_x axis by an amount which gives the prescribed current density J_x in the layer.

The Hall voltage (8) can be expanded in a series whose leading terms are

$$V_H=c_1 J_x B_y+c_2 B_y{}^2, \tag{9}$$

where $J_x=-Nev_d$, and v_d is the electron drift velocity in the x direction. Note that J_x has the dimensions current per unit length. An approximate value for the coefficient c_1 is obtained from the foregoing results if we replace the energy denominator D in (7) by the energy separation E_{10} between the bottoms of the lowest and first excited subbands. We find

$$c_1=-(2e/\epsilon)(z^2{}_{00}-z_{00}{}^2)/E_{10}. \tag{10}$$

For comparison, we can estimate a "classical" value for c_1 by neglecting the difference between Hall and drift mobilities, assuming the layer to have thickness z_{00},[5] and making the unwarranted[2,6] assumption that the Hall field is uniform throughout the layer. Then we find that

$$c_{1,\,\mathrm{cl.}}\sim -z_{00}/Ne. \tag{11}$$

The coefficient c_1 has the same dimensions as the conventional Hall coefficient, since N/z_{00} has dimensions of carriers per unit volume. However, for an unsymmetrical layer like an inversion layer, the term containing c_2 in (9) can make a significant contribution to the Hall voltage. This term is independent of J_x.

Values for the quantities which appear in Eqs. (10) and (11) have been calculated for inversion layers with several values of electron concentration N on silicon (100) surfaces with two different acceptor concentrations N_A. We use a dielectric constant of 11.8, and transverse and longitudinal masses of the conduction-band minima [corresponding to $m_1=m_2$ and to m_3, respectively, in Eq. (1)] equal to $0.19m$ and $0.98m$, respectively. The values of z_{00}, $z^2{}_{00}$, and E_{10} were obtained from numerical self-consistent solutions of the Schrödinger equation (1) and Poisson's equation for the quantized states at absolute zero with no magnetic field. The results are shown in Table I.

The perturbation-theory results in Eqs. (5) and (6) are not exact even to the order shown because they have not taken into account the change in the potential $V(z)$ in (1) which results from the magnetic-field–induced deformation of the states. To obtain fully self-consistent results which take

Table I. Perturbation-theory results for silicon inversion layers. N_A is the bulk-acceptor concentration, and N is the inversion-layer electron concentration. E_{10} is the energy separation between the bottoms of the lowest and the first excited subbands, and z_{00} and $z^2{}_{00}$ are the expectation values of z and z^2 in the lowest subband; they are obtained from self-consistent field calculations at zero magnetic field and zero absolute temperature. The Hall coefficient c_1 is obtained via Eqs. (9)-(11).

N_A (10^{20} m^{-3})	N (m^{-2})	E_{10} (meV)	z_{00} (Å)	$z^2{}_{00}/z_{00}{}^2$	c_1 (m^3/C)	$c_{1,\mathrm{class}}$ (m^3/C)
1	1×10^{14}	4.31	60.9	1.204	-3.4×10^{-5}	-3.8×10^{-4}
1	1×10^{15}	5.17	50.5	1.229	-2.2×10^{-5}	-3.2×10^{-5}
1	3×10^{15}	6.56	40.6	1.251	-1.2×10^{-5}	-8.4×10^{-6}
1	1×10^{16}	9.83	29.4	1.274	-4.6×10^{-6}	-1.8×10^{-6}
4	1×10^{14}	6.74	49.1	1.202	-1.4×10^{-5}	-3.1×10^{-4}
4	1×10^{15}	7.50	43.8	1.218	-1.1×10^{-5}	-2.7×10^{-5}

this into account, the Schrödinger equation and Poisson's equation were solved in the presence of a magnetic induction B_y. We obtained Δz by averaging over the occupied values of $\vec{k}$ as described in the paragraph following (8) and calculated the Hall voltage from Eq. (8). The coefficients c_1 and c_2 in (9), and the coefficient e_2 which gives the effective-mass change

$$m_1(0)/m_1(B_y)=1-e_2B_y{}^2, \tag{12}$$

were obtained from the fully self-consistent calculations and are given in Table II.

The approximate results for c_1 in Table I are seen to agree fairly well with the more accurate results in Table II. Values of e_2 obtained from Eq. (5) using the same approximations as those that lead to Eq. (10) agree quite well with the fully self-consistent values in Table II.

As a numerical example, we consider silicon with bulk acceptor concentration $N_A=10^{20}/\mathrm{m}^3$ (which corresponds to a room-temperature resistivity of about 125 Ω cm), inversion-layer electron concentration $N=10^{15}/\mathrm{m}^2$, a magnetic induction of 20 kG = 2 Wb/m^2, and current densities of +1 and −1 A/m. The calculated average values of z for these two cases are 50.615 and 50.238 Å, respectively, compared with 50.486 Å in zero magnetic field, and the respective Hall voltages are −20 and +38 μV. The effective mass in the k_x direction is increased by about a factor 1.0075. At higher magnetic fields, the effective mass varies appreciably with k_x, indicating the importance of higher order terms in Eq. (5) as well as in the other expansions given here.

There do not appear to be any measurements on Si inversion layers which can be directly compared with our results. The conductance of such layers for the magnetic field and current configuration considered here has been measured by Tansal, Fowler, and Cotellessa.[7] The changes they find are much too large to be accounted for by a simple addition of the Hall voltage as estimated here to the gate voltage or the substrate bias. Their results, like ours, show both a $\vec{J}\times\vec{B}$ term and a B^2 term. Additional contributions to

Table II. Results of fully self-consistent calculations. The values were obtained by fitting results for a range of values of magnetic induction and current density. They are expected to be correct within about 10% for $B_y \leq 3$ Wb/m^2 and for current densities corresponding to drift velocities $<10^4$ m/sec. The symbols not in Table I are c_2, from Eq. (9), and e_2, from Eq. (12).

N_A (10^{20} m^{-3})	N (m^{-2})	c_1 (m^3/C)	c_2 (V m^4/Wb^2)	e_2 (m^4/Wb^2)
1	1×10^{14}	-3.0×10^{-5}	5.6×10^{-7}	3.2×10^{-3}
1	1×10^{15}	-1.4×10^{-5}	2.3×10^{-6}	1.9×10^{-3}
1	3×10^{15}	-6.4×10^{-6}	1.9×10^{-6}	1.0×10^{-3}
1	1×10^{16}	-1.9×10^{-6}	0.6×10^{-6}	3.5×10^{-4}
4	1×10^{14}	-1.3×10^{-5}	1.9×10^{-7}	1.2×10^{-3}
4	1×10^{15}	-8.1×10^{-6}	1.1×10^{-6}	1.0×10^{-3}

the Hall voltage can come from the depletion layer,[8] since any carriers in that layer will have drift velocities comparable with those in the inversion layer and can be treated classically. At very low temperatures the number of such electrons is extremely small, and the considerations of Gurvich[9] may apply. The characteristic time for establishing a steady-state Hall voltage across the depletion layer is the dielectric relaxation time for the layer, which can be long enough to make possible an experimental separation of the slow depletion-layer contribution and the fast inversion-layer contribution.

At temperatures or inversion-layer concentrations high enough that more than one electronic subband is appreciably populated,[10] transfer of carriers between the subbands becomes important, and the simple considerations of this paper are no longer sufficient. Work is underway to extend the present results beyond the electric quantum limit, and to more general current, magnetic field, and surface orientations. When many subbands are occupied, the conventional theory[1,2] should be applicable.

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[10]This applies to most of the results of Ref. 7.

SPECTROSCOPIC DETERMINATION OF THE MAGNON DENSITY OF STATES OF $GdCl_3$, A FERROMAGNETIC RARE-EARTH SALT*

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The magnon density of states of ferromagnetic $GdCl_3$ in an external magnetic field has been observed by means of high-resolution optical spectroscopy. The relevant transitions arise from a single-ion transition mechanism, but the initial (magnon) and in some cases the final (exciton) states of the transitions are shown to have measurable dispersion, contrary to the usual description of rare-earth salts.

We have observed in the optical absorption spectrum of ferromagnetic $GdCl_3$, in an external magnetic field, transitions originating from a singly excited magnon whose line shapes are sufficiently resolved to yield information about the magnon density of states. To our knowledge, this is the first spectroscopic observation of the magnon dispersion of either a ferromagnet or a rare-earth salt. As a result, it is evident that high-resolution optical spectroscopy when used in conjunction with external magnetic fields can be useful in studying the magnon structure of materials with weak magnetic interactions.

In recent years magnons have been observed in the optical spectra of several antiferromagnetic transition-metal compounds[1] for which the magnetic interactions are one to two orders of magnitude stronger than that of the rare earths. These transitions always involve an exchange-coupled mechanism in which the magnon and exciton are created or destroyed simultaneously on different sites of the crystal. They therefore appear as spin-assisted sidebands to the pure electronic transitions of the metallic ion. On the contrary, the transitions originating from a magnon state of $GdCl_3$ arise from a single-ion transition mechanism. However, the excited states of the system can not always be classified simply as single-ion