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THEORY OF A JOSEPHSON OSCILLATOR*

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A quantum mechanical theory of superconducting tunnel junctions including noise is developed. The theory is applied to the calculation of the frequency pulling, linewidth of the radiation, and voltage power spectrum in the ac Josephson effect.

The tunneling of Cooper pairs between superconductors was first discussed by Josephson.[1] He derived the phenomenological equations governing the behavior of junctions, and this classical description is quite adequate to describe a large variety of phenomena. However, for a discussion of noise and related problems it is necessary to extend the theory. In this Letter we discuss a quantum mechanical theory of tunneling including noise and apply it to the calculation of the frequency pulling, linewidth, and voltage power spectrum in the ac Josephson effect.

We use a model in which Cooper pairs are added to the superconductor forming the left-hand side of the tunnel junction from a large normal electron reservoir. Cooper pairs are removed from the right-hand superconductor at the same rate by another normal electron reservoir. This approximates driving the tunnel junction from a constant-current source which is the usual experimental arrangement. The connection between the normal electron reservoir and the superconductor is described by a Hamiltonian

$$H_1=\sum_{k\alpha}g(b_k^+d_\alpha+d_\alpha^+b_k), \tag{1}$$

where $b_k^+=c_{k\uparrow}^+c_{-k\downarrow}^+$ creates a Cooper pair in the left-hand superconductor, d_α annihilates two electrons in the normal reservoir, and g is a constant. There is a similar term for the right-hand superconductor.

The equations of motion of the operators describing the superconductors are obtained by eliminating the reservoirs to second order in perturbation theory. This procedure, as well as the derivation of the noise sources in the equations of motion, has been discussed by Lax[2] and others. We will give only the results of these calculations. We suppose that the temperature is sufficiently low so that there are no normal excitations in the superconductors, and we consider only the paired states lying in the region $\mu\pm\hbar\omega_D$ around the Fermi energy. For simplicity in presentation here, we will neglect the kinetic energy of the paired states and adopt the strong-coupling model of a superconductor.[3] This only introduces certain qualitative changes and does not alter the final results. We then introduce the operators for the left-hand superconductor:

$$R=\sum_k b_k,\quad R^+=\sum_k b_k^+,$$
$$R_z=\tfrac{1}{2}\sum_k(1-2b_k^+b_k). \tag{2}$$

These operators behave like spin operators. In particular the total charge on the superconductor is $2eR_z$, where e is the electron charge. R and R^+ are related to the off-diagonal long-range order in the superconductor. There is another set of operators S, S^+, and S_z for the right-hand superconductor.

After elimination of the reservoir coordinates, the equations of motion of these operators are

$$\dot{R}=-i[R,H_0]-2\alpha R_zR+2\beta^+RR_z+F_R, \tag{3}$$

$$\dot{R}^+=-i[R^+,H_0]-2\alpha^+R^+R_z+2\beta R_zR^++F_{R^+}, \tag{4}$$

$$\dot{R}_z=-i[R_z,H_0]+(\alpha+\alpha^+)R^+R-(\beta+\beta^+)RR^++F_{R_z}, \tag{5}$$

where H_0 is the Hamiltonian of the superconductor and

$$\alpha = \sum_\alpha g^2 \int_0^\infty du \langle d_\alpha d_\alpha^+(-u)\rangle e^{2i\mu u}$$

$$\beta = \sum_\alpha g^2 \int_0^\infty du \langle d_\alpha^+ d_\alpha(-u)\rangle e^{-2i\mu u}, \qquad (6)$$

and μ is the chemical potential of the superconductor. The F represent noise sources whose properties can be calculated from the equations of motion of operators quadratic in the R, R^+, and R_z. The most important result required here is that

$$\langle F_{R_z}(t_1) F_{R_z}(t_2)\rangle = [(\alpha+\alpha^+)R^+R + (\beta+\beta^+)RR^+]\,\delta(t_1-t_2). \qquad (7)$$

When there is a voltage across the superconductors, pairs tunnel with the emission of a photon. The photons are emitted into a mode of the superconducting cavity which we describe by the operator b, b^+. The tunneling is described by a Hamiltonian

$$H_T = T(RS^+b^+ + SR^+b), \qquad (8)$$

where we have made the usual rotating-wave approximation. Introducing (8) into (3)-(5) and combining the equation for R with the corresponding equation for S^+, we obtain our working equations

$$\partial(RS^+)/\partial t = [-i\omega + (\lambda_L + \lambda_R)R_z]RS^+ - 2iTbR_z(RR^+ + SS^+) + F_{RS^+}, \qquad (9)$$

$$\partial R_z/\partial t = -\lambda_L RR^+ - iT(RS^+b^+ - SR^+b) + F_{R_z}, \qquad (10)$$

$$\partial b/\partial t = (-i\Omega - \tfrac{1}{2}\gamma)b - iTRS^+ + f. \qquad (11)$$

In these equations we have retained only the real part λ_L of $2(\beta_L{}^+ - \alpha_L)$ (see below). We distinguish quantities on the left-hand side and right-hand side by the subscripts L and R. The Josephson frequency is $\hbar\omega = 2(\mu_L - \mu_R)$ and is related to the voltage across the barrier and the charge by

$$\omega = \frac{2eV}{\hbar} = -\frac{4e^2}{\hbar C}R_z, \qquad (12)$$

where C is the capacitance of the barrier. In (11), Ω is the passive cavity frequency and $\frac{1}{2}\gamma$ the decay constant of the cavity. The properties of the noise sources f for the cavity have been given by Lax[2]:

$$\langle f(t_1)f^+(t_2)\rangle = \gamma(\bar{n}+1)\delta(t_1-t_2),$$
$$\langle f(t_1)f(t_2)\rangle = \gamma\bar{n}\delta(t_1-t_2), \qquad (13)$$

where $\bar{n}$ is the density of blackbody radiation at the frequency Ω. In the present case we set $\bar{n}=0$. We need not consider the equation for S_z as charge is conserved, $R_z + S_z = 0$.

The operators in (9)-(11) are considered to be mean values so that their order is unimportant. We get the steady state solution of (9)-(11) by neglecting the noise sources and assuming a time dependence of b and RS^+ of $e^{-i\nu t}$. Using the subscript 0 for steady-state values from (11),

$$b_0 = \frac{iTR_0S_0{}^+}{i(\nu-\Omega) - \frac{1}{2}\gamma}. \qquad (14)$$

Substituting this in (9) and equating imaginary and real parts to zero gives two equations. The first determines the frequency ν:

$$\nu = \frac{\frac{1}{2}\gamma\omega + \Gamma\Omega}{\frac{1}{2}\gamma + \Gamma}, \qquad (15)$$

$$\Gamma = \frac{T^2\gamma|R_{z0}|(R_0R_0{}^+ + S_0S_0{}^+)}{(\nu-\Omega)^2 + \frac{1}{4}\gamma^2}. \qquad (16)$$

The second equation determines the dc current-voltage characteristic and after some rearrangement is

$$J = \frac{2eT^2\gamma|R_0S_0{}^+|^2}{(\nu-\Omega)^2 + \frac{1}{4}\gamma^2} = 2e\gamma|b_0|^2. \qquad (17)$$

The last form in (17) is consistent with energy conservation. Equation (15) determines the frequency pulling. To estimate Γ we write it in the form

$$\Gamma = \frac{J}{2e}\,\frac{(R_0R_0{}^+ + S_0S_0{}^+)|R_{z0}|}{|R_0S_0{}^+|^2}. \qquad (18)$$

From the BCS theory $R_0 = 2N(0)\Delta_g \ln 2\omega_D/\Delta_g$, where Δ_g is the energy gap and $N(0)$ is the density of states. For a superconducting volume of 10^{-6} cm^3, $R_0 \simeq 10^{12}$. Then using $J = 10$ mA, $V = 20$ μV, and $C = 10^4$ cm, we find $\Gamma \simeq 10^{-1}$ sec^{-1}. A typical cavity γ is 10^8 sec^{-1}; so the frequency pulling is small.

Assuming that we are well above threshold, we linerarize (9)-(11) about the steady operating point by writing

$$b=b_0e^{-i\nu t}e^{u+i\varphi},$$

$$RS^+=R_0S_0^+e^{-i\nu t}e^{x+i\theta},\quad \text{and}\quad R_z=R_{z0}e^y.$$

A similar procedure has been used by Lax[4] in connection with lasers. The change x in the amplitude of RS^+ is very small and will be neglected. It is essential to retain the coupling between y, θ, and φ as they are strongly coupled through the term ω in (9); i.e., the oscillator is frequency modulated. This small-oscillation analysis shows that there is one neutral mode of oscillation corresponding to changing the overall phase of the coupled systems. The system is stable provided the detuning $\Delta=\Omega-\nu>0$. At $\nu=\Omega$ it becomes unstable. Eliminating all the variables except φ from the small-oscillation equations, we find to a good approximation that

$$\frac{\partial\varphi}{\partial t}=\operatorname{Im}\frac{f}{b_0}+\frac{1}{\gamma\Delta}\left(\frac{\gamma^2}{4}-\Delta^2\right)\operatorname{Re}\frac{f}{b_0}+\frac{4e^2}{\hbar}R_DF_{R_z}. \tag{19}$$

Using the properties of the noise sources, the radiation linewidth due to phase diffusion after some rearrangement is

$$D=\frac{\langle[\varphi(t)-\varphi(0)]^2\rangle}{t}=\frac{16e^3}{\hbar^2}JR_D^{\,2}, \tag{20}$$

where R_D is the dynamical resistance $(\partial J/\partial V)^{-1}$. The line is Lorentzian in shape and D is the full width at half-maximum. In this model there is an equal contribution to D from f and F_{R_z} in (19).

The physical origin of this linewidth lies in the shot noise associated with photons leaving the cavity. This shot noise is also reflected in the pair tunneling current and gives rise to voltage fluctuations. Thus for low frequencies, from the small-oscillation analysis, the spectrum of voltage fluctuations is

$$\langle V_\omega V_{-\omega}\rangle=4eJR_D^{\,2}/(1+\omega^2\tau^2), \tag{21}$$

where

$$\tau^2=(R_DC+\gamma^{-1})^2-\frac{2\gamma R_DC}{\Delta^2+\frac{1}{4}\gamma^2},$$

and is characteristic of shot noise. There is no shot noise of the form (21) in the dc Josephson effect. Using the power spectrum (21) and the fact that the oscillator is frequency modulated[5] leads exactly to (20).

At finite temperatures the contribution to D from the cavity shot noise is multiplied by $(2\bar{n}+1)$, where $\bar{n}$ is the number of blackbody photons in the cavity at the cavity-wall temperature. Thus at finite temperatures

$$D=\frac{8e^3}{\hbar^2}JR_D^{\,2}\left(\coth\frac{eV}{kT}+1\right). \tag{22}$$

This expression is in good agreement with experiment. For $J=5$ mA, $R_D=1$ mΩ, $V=20$ μV, and $T=1.5$°K, which are values appropriate for a Sn junction, we find $D/2\pi=19$ kc/sec. The experimental value is about 15 kc/sec.[6] Also D_0/R_D has a minimum value $4e^2\gamma/\hbar$ when the detuning $\Delta=\frac{1}{2}\gamma$. R_D is approximately a minimum here. This is also in accord with the experimental data. Close to T_c it is necessary to include the voltage fluctuations due to the normal component of the tunneling current and the linewidth increases rapidly. This will be discussed elsewhere.

The small-oscillation analysis also gives information on the intensity fluctuations of the radiation. If $p=b^+b-|b_0|^2$, then it is found for $\Delta=\frac{1}{2}\gamma$ that

$$\langle p^2\rangle=|b_0|^2 \quad \text{if} \quad \gamma R_DC\gg 1,$$

$$=\frac{2e^2}{\hbar C\gamma}|b_0|^4 \quad \text{if} \quad \gamma R_DC\ll 1. \tag{23}$$

In the first case the radiation intensity has a second moment appropriate to coherent radiation, and in the second case the distribution is broadened.

Finally it should be mentioned that a frequency shift can occur in (9) through the imaginary part of $(\beta^+-\alpha)$. This is a typical Lamb shift due to the reservoir interaction and is proportional to the voltage. We estimate this shift from (6) by assuming the normal electron reservoir is a free-electron gas and taking $d_\alpha=a_{q_1\uparrow}a_{q_2\downarrow}$ where the a are free-electron annihilation operators. The sums over q are cut off in energy at $\mu\pm\hbar\omega_D$. A simple calculation gives a frequency shift of $(4\Gamma/\pi)\ln 2$ which is a small. A more complete account of this work will be published elsewhere.

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NONLINEAR AMPLIFICATION AND AUTOMODULATION OF SOUND IN PIEZOELECTRIC SEMICONDUCTORS

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We develop a quantitative nonlinear theory of sound amplification by carrier drift in piezoelectric semiconductors and predict an effect of sound automodulation, i.e., quasi-periodical variation of the intensity of monochromatic sound due to the nonlinear interaction of the sound with its own acoustoelectric field.

We have developed a quantitative small-signal nonlinear theory of sound amplification by carrier drift in homogenous piezoelectric semiconductors. The form of the stationary distribution of the sound intensity along the sample is investigated. Under certain circumstances the stationary distribution becomes unstable, and as the result, automodulation of the ultrasonic signal propagating through the sample may take place; i.e., the sound intensity begins to oscillate. The conditions for onset of the oscillation are investigated. We give here a brief account of the results obtained. The details of our calculation will be published elsewhere.

Due to nonlinear interactions the ultrasonic attenuation (or amplification) becomes dependent on the sound intensity I. There are two sources of such dependence. First, the amplification constant at any point of the sample depends on the sound intensity at the same point. Second, it depends upon the dc field E at this point. To first approximation in the sound intensity I the expression for the attenuation constant is

$$\Gamma = \Gamma_0 + \Gamma' I + (\partial\Gamma_0/\partial E_0)(E - E_0). \quad (1)$$

Here Γ_0 is the linear-attenuation constant, $E_0 = U/L$, L is the sample length, and U is the voltage along sample which is supposed to be fixed, independent of the sound intensity.

The total current through any cross section of the sample should be constant as the consequence of the charge conservation. The total current density j is the sum of the conduction current $\sigma_0 E$ (where σ_0 is the conductivity in the absence of sound) and of the acoustoelectric current $j^{ac} = FI$, which is proportional in the first approximation to the sound intensity. (The sound being amplified, its intensity is coordinate dependent.) The electric field should be redistributed along the sample to maintain the constancy of the total current. The dependence of the electric field on the sound intensity can be derived from the equation

$$j = \sigma_0 E + FI = \sigma_0 E_0 + (F/L)\int_0^L I(x')dx'. \quad (2)$$

Thus the expression for the nonlinear-attenuation constant including the terms of first order in sound intensity has the form

$$\Gamma = \Gamma_0 + \Gamma_1 I + \Gamma_2 (1/L)\int_0^L I(x)dx, \quad (3)$$

where

$$\Gamma_1 = \Gamma' - \Gamma_2; \quad \Gamma_2 = \frac{F}{\sigma_0}\frac{\partial\Gamma_0}{\partial E_0}.$$

The second and the third terms in Eq. (3) will be referred to as the local nonlinear term and the nonlocal nonlinear term, respectively. As has been pointed out by many authors,[1] the nonlocal nonlinear interaction acts as a feedback mechanism.[2] On account of this mechanism the intensity of sound at the points that are nearer to the input end of the crystal depends on the intensity of sound at more distant points.

The expression (1) for the nonlinear attenuation constant was previously obtained by Gurevich and Laikhtman[3] and Katilius.[4] They concluded that the constant Γ_1 as well as Γ_2, depending on relative role of different nonlinearities, particularly those produced by traps, can be of any sign, and their values may vary over a wide range. It is