PROPERTIES OF HIGH-DENSITY MATTER IN INTENSE MAGNETIC FIELDS

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We summarize recent results on the properties of matter in intense magnetic fields which are believed to exist in gravitationally collapsed bodies. The equations of state of a Fermi gas in magnetic fields are presented; they are superpositions of quantities relevant to a one-dimensional gas if the density is low enough or the field strong enough. For laboratory matter, the transition field is approximately 10^9 G.

According to the flux-conservation law of classical electrodynamics, in a contraction process the magnetic field associated with a current-carrying plasma will increase as a^{-2}, where a is a scaling factor <1. Magnetic fields exist in stars. The average magnetic field of the sun is 1 G, but in sun spots the field strength may exceed several thousand gauss. Stars with magnetic fields greater than 500 G (which is the present lower limit of detectability) also exist abundantly in nature, with the highest field ever observed being in the neighborhood of 3×10^4 G.

During gravitational collapse of a star, a contraction from an initial radius of an order of 10^{11} cm to the Schwarzschild radius of an order of 10^6 cm can take place. This means that the magnetic field of collapsed objects can be increased by a factor of 10^{10} or more. Even in a nonmagnetic star like the sun a field strength of the order of 10^{13} G may be achieved locally after gravitational collapse. Thus, we expect fields of the order or greater than 10^{13} G to exist in gravitationally collapsed bodies.

In this Letter we shall discuss the properties of matter in intense magnetic fields only, without going into the more fundamental problem of the origin of intense magnetic fields. The solution to the Dirac equation of a free electron in an external magnetic field has been obtained four decades ago.[1] The energy states $E_\pm(p_z, n, r, H)$ of the electron (+) and the positron (−) are given by the equation

$$E_\pm(p_z,n,r,H) = \pm mc^2[1+(p_z/mc)^2 + 2(H/H_c)(n+r-1)]^{1/2}, \qquad (1)$$

where p_z is the z momentum and H is the field which is in the z direction, $H_c = m^2c^3/e\hbar = 4.414 \times 10^{13}$ G, and

$$n=0,1,2,\cdots;\quad r=1,2; \qquad (2)$$

n is the principal quantum number and r is the spin quantum number. In the weak-field or high-energy limit the transition to a classical electron is achieved by the transition

$$2(H/H_c)(n+r-1)\to(p_x^{\,2}+p_y^{\,2})/(mc)^2;$$
$$\textstyle\sum_n \to \int dn, \qquad (3)$$

where p_x and p_y are the x and y momenta, respectively.

According to Eq. (1) the lowest energy states of the electron and the positron are $\pm mc^2$, respectively; this means that the presence of a magnetic field does not alter the separation energy between the positive and negative energy states. Thus, in a strong magnetic field, even when the classical spin energy $\mu_B H$ exceeds mc^2 (μ_B is the Bohr magneton), spontaneous pair creation in an external field constant in time and homogeneous in space cannot take place.

Because of anisotropy of the system the usual thermodynamic pressure must be replaced by the stress tensor. If we neglect interactions then there is no shear. The stress tensor is then diagonal in Cartesian coordinates, though not isotropic. In the frame of reference in which the system is at rest, the off-diagonal time elements (which represent macroscopic momenta of the system) also vanish. The nonvanishing diagonal elements P_{xx}, P_{yy}, and P_{zz} (normal stresses in the x, y, and z direction, respectively) and the energy density U (including the rest energy of the electrons) are given by the following expressions[2,3]:

$$P_{xx}=P_{yy}=P_0\left(\frac{H}{H_c}\right)^2\sum_{n=1}^{\infty} nC_1\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right), \qquad (4)$$

$$P_{zz}=P_0\left(\frac{H}{H_c}\right)\left[\tfrac{1}{2}C_2(\varphi,\mu)+\sum_{n=1}^{\infty}a_n^{\,2}C_2\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right)\right], \tag{5}$$

$$U=P_0\left(\frac{H}{H_c}\right)\left[\tfrac{1}{2}C_3(\varphi,\mu)+\sum_{n=1}^{\infty}a_n^{\,2}C_3\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right)\right], \tag{6}$$

and the particle number density N and the magnetic moment density M are given by[4,5]

$$N=N_0\left(\frac{H}{H_c}\right)\left[\tfrac{1}{2}C_4(\varphi,\mu)+\sum_{n=1}^{\infty}a_nC_4\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right)\right], \tag{7}$$

$$M=M_0\left[\tfrac{1}{2}C_2(\varphi,\mu)+\sum_{n=1}^{\infty}a_n^{\,2}C_2\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right)-\left(\frac{H}{H_c}\right)\sum_{n=1}^{\infty}nC_1\left(\frac{\varphi}{a_n},\frac{\mu}{a_n}\right)\right]$$

$$=\left[\frac{P_{zz}}{P_0(H/H_c)}\right]M_0\left[1-\frac{P_{xx}}{P_{zz}}\right], \tag{8}$$

where $P_0=\pi^{-2}mc^2\lambda_C^{-3}$, $\lambda_C=\hbar/mc$, $N_0=\pi^{-2}\lambda_C^{-3}$, $M_0=2\mu_BN_0$, $a_n=[1+2(H/H_c)n]^{1/2}$, $\varphi=kT/mc^2=T/5.903\times10^9$ °K (T is the temperature in degrees Kelvin), $\mu=(\tilde{\mu}+mc^2)/mc^2$ and $\tilde{\mu}$ is the chemical potential of the electron. The rest of the symbols have their usual meaning. The functions C_k are expectation values of a one-dimensional Fermi gas, with

$$C_1(\varphi,\mu)=\langle\epsilon^{-1}\rangle;\quad C_2(\varphi,\mu)=\langle v(dv/d\epsilon)\rangle;\quad C_3(\varphi,\mu)=\langle\epsilon\rangle;\quad C_4(\varphi,\mu)=\langle 1\rangle, \tag{9}$$

where

$$\langle u\rangle=\int_0^{\infty}u\left[1+\exp\left(\frac{\epsilon-\mu}{\varphi}\right)\right]^{-1}dv;\quad \epsilon=(1+v^2)^{1/2}. \tag{10}$$

v may be regarded as the momentum and ϵ the total energy of a one-dimensional particle of unit mass. $C_2(\varphi,\mu)$ is the pressure, $C_3(\varphi,\mu)$ the average energy, and $C_4(\varphi,\mu)$ the particle density of a one-dimensional noninteracting Fermi gas at a temperature T with a chemical potential μ.

It can be shown that in the limit of large values of n [Eq. (3)], $P_{xx}\to P_{zz}$ and P_{xx}, P_{zz}, U, and N all reduce to corresponding quantities for a noninteracting Fermi gas[2,3]; the expression for M reduces to the Curie-Langevin law in the nondegenerate and nonrelativistic limit.[4,5] Thus, the properties of a Fermi gas are strongly altered by a magnetic field if

$$\langle(p_z/mc)^2\rangle\leq 2(H/H_c). \tag{11}$$

Figure 1 shows the temperature-density domains of a magnetized Fermi gas for a number of field strengths.

Fermi degeneracy also takes place in the limit $T\to0$. However, an important case is when $T\to0$ and $\mu\leq[1+2(H/H_c)]^{1/2}$. It can be shown that in this limit,[3]

$$P_{xx}=P_{yy}=0,$$
$$P_{zz}=\tfrac{1}{2}P_0(H/H_c)C_2(0,\mu),$$
$$U=\tfrac{1}{2}P_0(H/H_c)C_3(0,\mu),$$
$$N=\tfrac{1}{2}N_0(H/H_c)C_4(0,\mu), \tag{12}$$

where for $\mu\geq1$,

$$C_2(0,\mu)=\tfrac{1}{2}\{\mu(\mu^2-1)^{1/2}-\ln[\mu+(\mu^2-1)^{1/2}]\},$$
$$C_3(0,\mu)=\tfrac{1}{2}\{\mu(\mu^2-1)^{1/2}+\ln[\mu+(\mu^2-1)^{1/2}]\},$$
$$C_4(0,\mu)=(\mu^2-1)^{1/2}, \tag{13}$$

and for $\mu\leq1$, $C_k(0,\mu)=0$. Thus, in this limit, the lateral stress of the gas vanishes and the gas becomes exactly a one-dimensional gas discussed extensively elsewhere[6] [see also the comment after Eq. (10)]. The field strength needed for transition into a one-dimensional gas for laboratory

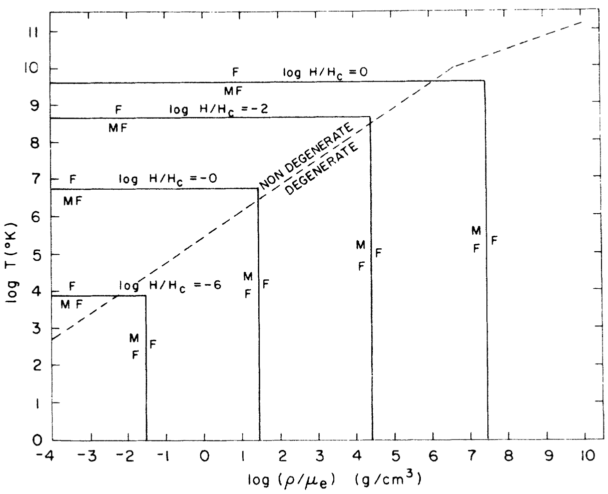

FIG. 1. Approximate domains of a magnetized Fermi gas for some field strengths, as indicated by the solid lines. The magnetized gas domains are marked "*MF*."

matter with $N=10^{23}$ cm^{-3} is of the order of 10^9 G. At present, the strongest field that can be generated in laboratory is of the order of 10^6 G. It is hoped that future technology will allow laboratory generation of fields strong enough to create one-dimensional matter, which should be of interest not only to astrophysicists, but also to solid-state physicists.

Figure 2 shows the relations between several thermodynamic functions. It is seen that most thermodynamic functions have discontinuous derivatives; this behavior is caused by the irregular behavior of the density of states in a magnetic field near the transition energy $E_n=[(1+2nH/H_c)^{1/2}-1]mc^2$.[7] There are also large regions in which $P_{zz}/N^{4/3}$ decreases with N. It is well known that in a star with spherical symmetry, when $P/N^{4/3}$ decreases with N (at zero temperature) dynamical instability will take place, and gravitational collapse will result.[8] In our case the asymmetry is strongly correlated with field-line configurations so that no conclusive statement regarding dynamical stability may be made unless the field configuration is known.

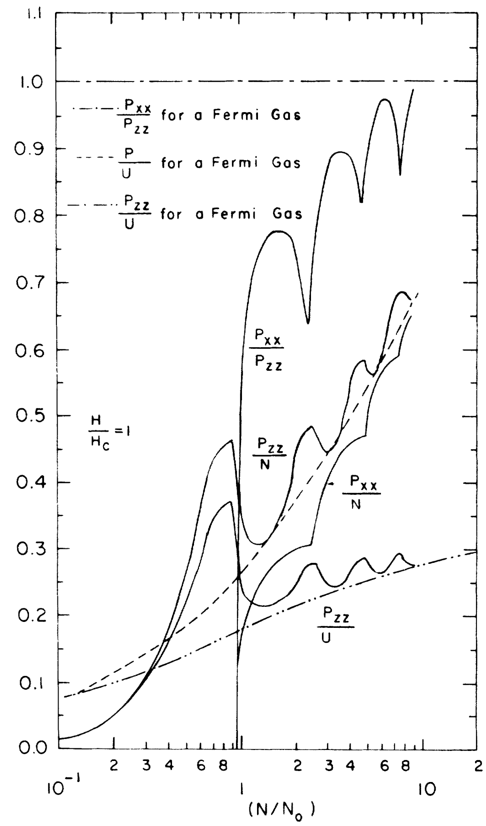

FIG. 2. Functional dependence of P_{xx}/P_{zz}, P_{zz}/N, P_{xx}/N, and P_{zz}/U on N for the degenerate case with $H/H_c=1$. The corresponding functions for a Fermi gas in the absence of a magnetic field are also shown for comparison.

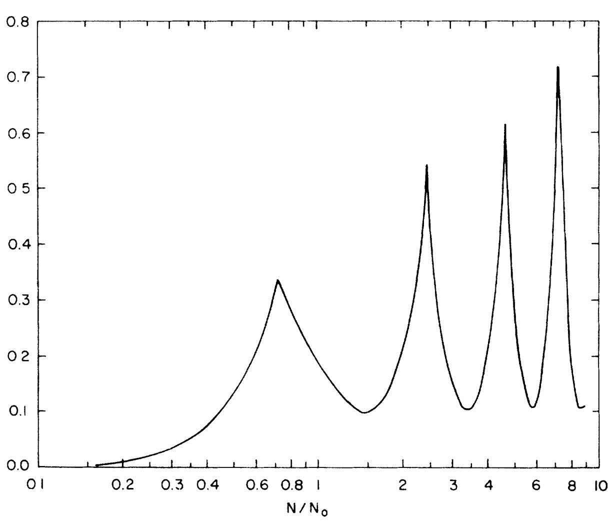

FIG. 3. The relation between M and N/N_0 for $H/H_c=1$. The peaked behavior of M is related to the de Haas–van Alphen effect.

Figure 3 shows the behavior of M as a function of N. M is never negative and shows sharp peaks at certain values of N. However, the maximum value of induced M never exceeds a few times $(\mu_B/\lambda_C^3)(H/H_c)\approx 10^{-3}H$. The condition for permanent magnetism to exist is $M \geqslant H$. Thus we can conclude that permanent magnetism is absent in a noninteracting electron gas at any temperature and at any density.

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MASS SPECTRUM FOR $\pi^-\pi^-\pi^+$ PRODUCED IN π^-p AT BeV/c†

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The negative three-pion mass spectrum between 1.0 and 1.4 BeV has been examined in the reactions $\pi^-p \to p(\pi^-\pi^-\pi^+)$, $\pi^-p \to p\pi^0(\pi^-\pi^-\pi^+)$, and $\pi^-p \to n\pi^+(\pi^-\pi^-\pi^+)$ for 5-BeV/c incident π^-. In each reaction peaks are observed near 1.06, 1.17, and 1.30 BeV.

In the charged three-pion mass spectrum produced in $\pi^\pm p$ interactions several peaks have been reported in the region between 1 and 1.4 BeV. Of these the A_2 peak near 1.3 BeV is most certain, with other decay modes such as $K\bar{K}$ and $\eta\pi$ reasonably established.[1] The region below the A_2 is less certain. Although early experiments appeared to resolve an A_1 peak near 1.06 BeV from the A_2, later contributions with higher statistics have made the conclusions uncertain.[1] The largest samples of data have come from the reactions $\pi^\mp p \to (\pi^\mp\pi^-\pi^+)$, where a large background is expected from a Deck-type mechanism process with ρ^0 production.[2] In other reactions such as $\pi^-p \to p\pi^0(\pi^-\pi^-\pi^+)$ and $\pi^-p \to n\pi^+(\pi^-\pi^-\pi^+)$ in which the simple Deck mechanism is not possible, the background should be different and allow an independent examination for resonances below the A_2.

We have studied the reactions

$$\pi^- p \to p(\pi^-\pi^+\pi^-), \tag{1}$$

$$\pi^- p \to p\pi^0(\pi^-\pi^+\pi^-), \tag{2}$$

$$\pi^- p \to n\pi^+(\pi^-\pi^+\pi^-), \tag{3}$$

in π^-p interactions at 5 BeV/c. The results were obtained in an exposure of 150 000 pictures in the Berkeley 72-in. bubble chamber and roughly 55 000 four-prong events have been measured. From these events 6300 fits were obtained in Reactions (1).

For Reactions (2) and (3) we required that the events have a χ^2 confidence level larger than 10%. We also required the square of the missing mass corresponding to the π^0 for Reaction (2) to be between -0.06 and 0.10 BeV^2, and the square of the missing mass corresponding to the neutron for Reaction (2) to be between 0.74 and 1.02 BeV^2. These fits were checked for consistency with ionization. In the case of further ambiguities the best χ^2 fit was used. This left 6822 and 6216 accepted fits for Reactions (2) and (3), respectively. In Reaction (2) there are strong peaks for N^{*++}, ω, and η which are incompatible with $\pi^-\pi^+\pi^-$ resonance formation and we have therefore exclude these events.

The $\pi^-\pi^+\pi^-$, $\rho^0\pi^-$, and $\pi^-\pi^+$ mass spectra for the three reactions are shown in Fig. 1. After the above selections have been made the $\pi^-\pi^+\pi^-$ and $\rho^0\pi^-$ mass spectra between 1.0 and 1.4 BeV are best interpreted in terms of three enhance-