SATURATION OF THE ISOSPIN-FACTORED CURRENT ALGEBRA AT INFINITE MOMENTUM

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The angular conditions for the isospin-factored current algebra at infinite momentum are decomposed into a set of momentum-transfer–independent conditions. The solutions to the angular conditions are constructed and related to wave equations. The existence of spacelike solutions is established for any nontrivial solution, and the decoupling problem is discussed.

Some time ago Dashen and Gell-Mann[1] proposed a concrete scheme for saturating the algebra of current density at $p_z=\infty$. A natural first step is to try to saturate the algebra of isospin charge densities by a set of states all having isospin $\frac{1}{2}$. This simplifies the mathematics, and may be even directly relevant for physics since all the strangeness $S=\pm1$ mesons and/or $S=\pm2$ baryons so far observed have isospin $\frac{1}{2}$.

When restricted to a set of $I=\frac{1}{2}$ states, the current may be written as $F_i(\vec{k})$, $i=1,2,3$, where τ is the usual isospin matrix and $\vec{k}$ is the two-dimensional transverse momentum transfer. In Ref. 1 it is found that the $j(\vec{k})$ must satisfy the conditions

$$j(\vec{k})=\exp(i\vec{k}\cdot\vec{X}), \tag{1}$$

where $\vec{X}=(X_1,X_2)$ are any two commuting self-adjoint operators, and

$$\{I,\{I,\{I,j(\vec{k})\}\}\}=\{I,\{J,j(\vec{k})\}\}, \tag{2}$$

where

$$\{I,\Theta\}=\tfrac{1}{2}[M^2,[J_3,\Theta]]-[\vec{k}\cdot M\vec{J},\Theta]-\tfrac{1}{2}k^2[J_3,\Theta]_+,$$
$$\{J,\Theta\}=\tfrac{1}{4}[M^2,[M^2,\Theta]]+\tfrac{1}{2}k^2[M^2,\Theta]+\tfrac{1}{4}k^4\Theta,$$

M is the mass operator, J_3 the helicity operator, and $\vec{J}=(J_1,J_2)$, where J_α, $\alpha=1,2,3$, generate the rotation group in helicity space:

$$[J_\alpha,M]=0,\quad [J_\alpha,J_\beta]=i\epsilon_{\alpha\beta\gamma}J_\gamma. \tag{3}$$

Equation (1) is enforced by the charge-density algebra and Eq. (2) by the condition that the saturation be Lorentz invariant. Equation (2) is simply another form of Eq. (11) of Ref. 1. Any solution of Eqs. (1)-(3) is a solution of the saturated algebra at $p_z=\infty$, with the mass spectrum given by M and the form factors by $\exp(i\vec{k}\cdot\vec{X})$. A number of particular solutions to these equations are already known.[2-4]

The purpose of the present note is to present the following results of a more general investigation of Eqs. (1)-(3):

(1) Elimination of the momentum transfer. —Multiplying Eq. (2) to the left by $\exp(-ik\cdot X)$ and expanding in powers of k, we find that if the expansion terminates,[5] and if we reject solutions for which a spacelike part ($M^2<0$) exists and is definitely coupled to the timelike part by the current, then Eqs. (1)-(2) are equivalent to the set of k-independent equations for M, J_3, $\vec{X}$, and $\vec{J}$ given in Table I.

(2) Primitive solutions.—By choosing

$$B=0,\quad B=\tfrac{1}{2}(R+1),\quad B=-\tfrac{1}{2}(R+1), \tag{4}$$

respectively, the equations of Table I can be reduced to three primitive (mutually exclusive) sets of equations. The primitive equations can be solved completely[6] and the solutions are as follows: For $B=0$, the six operators $\vec{X}$, J_3, $\vec{F}=\vec{\Lambda}+\frac{1}{2}[\vec{X},M^2]$, and $\bar{K}_3=K_3-\frac{1}{2}iR$ form an exact SL(2, C) algebra, and satisfy the pseudo-Hermitian condition $O^\dagger=(1-\epsilon)^{-1}O(1-\epsilon)$. The mass operator is given by $M^2=(1-\epsilon)^{-1}(g_0-g_3+s)$, where s is any scalar, g_μ is any vector, and $\epsilon=g_0+g_3$. M^2 and ϵ are Hermitian. For $B=\pm\frac{1}{2}(R+1)$, the six operators $\vec{X}$, $\bar{J}_3=J_3\mp\frac{1}{2}$, $\vec{F}=\vec{\Lambda}\pm\frac{1}{2}[M^2,[J_3,\vec{X}]]$, and K_3 form an exact SL(2, C) algebra and are Hermitian. The mass operator is given by $M^2=(g_0-g_3)+g_\mp(1-\epsilon)^{-1}g_\pm$, where g_μ is any Hermitian vector, $g_\pm=g_1\pm ig_2$, and $\epsilon=g_0+g_3$.

(3) Wave equations.—There is the following connection between the equations of Table I and infinite-component wave equations: The three primitive solutions may be derived from the wave equations

$$(p^2-2g\cdot p-s)\varphi=0,\quad (\gamma\cdot p-\gamma\cdot g)\psi=0,\quad \gamma_5\psi=\pm\psi, \tag{5}$$

respectively, where g is any p-independent vector and s is any p-independent scalar.

Nonprimitive solutions connecting the spaces $B=\pm\frac{1}{2}(R+1)$ can be derived from the wave equation

$$(\gamma p-\mathfrak{M})\psi=0, \tag{6}$$

Table I. Momentum-transfer–independent equations for M, J_3, $\vec{X}$, and $\vec{J}$. The independence of the equations can be checked by considering the special case M =const.

Definitions	Equations
$X_+ = X_1 \pm iX_2, \ J_\pm = J_1 \pm iJ_2$	$[X_i, [X_j, [X_k, M^2]]] = 0$
$M_\pm = [X_\pm, M^2]$	$[X_+, \Lambda_+] = [X_-, \Lambda_-] = 0$
$M_{\pm\pm} = [X_\pm, [X_\pm, M^2]]$	$[B, X_i] = 0, \ i[K_3, X_i] = X_i$
$\varepsilon = (R+1)^{-1}R, \ R = -\frac{1}{4}M_{+-}$	$[B, R] = 0, \ B[B^2 - \frac{1}{4}(R+1)^2] = 0$
$\Lambda_\pm = \pm 2iMJ_\pm + \frac{1}{2}[M^2, X_\pm]_+$	$M_{++}^2 = 0$
$iK_3 \pm (J_3 - B) = \frac{1}{4}\cdot[\Lambda_\mp, X_\pm]$	$M_{++}[B + \frac{1}{2}(R+1)] = [B - \frac{1}{2}(R+1)]M_{++} = 0$
$G = B^2 - \frac{1}{4}(R+1)^2$	$M_{++}G_+ = M_{--}G_- = 0$
$G_\pm = \pm\frac{i}{4}[K_3, \Lambda_\pm + \frac{1}{2}M_\pm] \pm \frac{1}{4}(\Lambda_\pm + \frac{1}{2}M_\pm)$	$M_+G - [\Lambda_+ + \frac{1}{2}M_+, G] + 4[B - \frac{1}{2}(R+1)]G_+ = 0$
	$M_-G - [\Lambda_- + \frac{1}{2}M_-, G] - 4[B + \frac{1}{2}(R+1)]G_- = 0$
	$2M_+G_+ - [\Lambda_+ + \frac{1}{2}M_+, G_+] + \frac{i}{8}[K_3, M_{++}M^2] + \frac{1}{4}M_{++}M^2 = 0$
	$2M_-G_- - [\Lambda_- + \frac{1}{2}M_-, G_-] - \frac{i}{8}[K_3, M_{--}M^2] + \frac{1}{4}M_{--}M^2 = 0$

where $\mathfrak{M}$ is any p-independent scalar[7] such that $[\gamma_5, \beta\mathfrak{M}] \neq 0$. Nonprimitive solutions connecting all three spaces $B = 0, B = \pm\frac{1}{2}(R+1)$ can be derived from the coupled wave equations

$$(\gamma\cdot p - \mathfrak{M})\psi = \lambda\varphi, \quad (p^2 - 2g\cdot p - s)\varphi = \bar{\lambda}\psi, \tag{7}$$

where λ is any p-independent spinor, and $\bar{\lambda} = (1-\epsilon)\lambda^\dagger\gamma_0$.

The fact that Eqs. (5) represent the most general solution to the primitive equations, and that they are only mildly p dependent, suggests that (7) may already represent the most general solution to the equations of Table I.

(4) Spacelike solutions.—Under very mild technical conditions,[8] the equations of Table I can be shown to be either physically trivial or admit spacelike as well as timelike solutions.[9] Proof: From the first set of equations in Table I, we have

$$(\exp(i\lambda X_k)d, M^2\exp(i\lambda X_k)d) = (d, M^2d) + i\lambda(d, [X_k, M^2]d) - \tfrac{1}{2}\lambda^2(d, [X_k, [X_k, M^2]]d), \tag{8}$$

whence

$$M^2 \geq 0 \to [X_k, [X_k, M^2]] \leq 0 \to R \geq 0 \to 0 \leq \epsilon < 1. \tag{9}$$

On the other hand, by suitably combining the equations of Table I with Eqs. (3) we obtain the relation

$$i[K_3, \epsilon] = \epsilon + \tfrac{1}{64}(1-\epsilon)[M_{++}, M_{--}]_+(1-\epsilon). \tag{10}$$

Hence, if we define the functions

$$\Phi(x) = (e^{ixK_3}d, \epsilon e^{ixK_3}d), \quad \|d\| = 1, \tag{11}$$

we have from (10) and the last inequality in (9)

$$\Phi'(x) \geq \Phi(x) \text{ and } 0 \leq \Phi(x) < 1, \quad -\infty < x < \infty, \tag{12}$$

respectively. But these inequalities are incompatible unless $\Phi(x) = 0$. Hence

$$M^2 \geq 0 \to \Phi(x) = 0 \to \epsilon = 0 \to R = 0 \to [X_k, [X_k, M^2]] = 0. \tag{13}$$

The last implication follows from the second inequality in (9). But then the coefficient of λ^2 in (9) vanishes. Hence, according as the coefficient of λ vanishes or not, M^2 commutes with the current operator, or is not positive.

The existence of the spacelike solutions obviously represents a serious difficulty for the program of saturating with timelike one-particle states. However, there still remains the question of coupling.

(5) Coupling of spacelike solutions.—It would be possible to saturate current algebra consistently with timelike solutions alone if the current did not couple them to the spacelike solutions. This happens for the free-quark solution, but not[3,10] for the two previously known nontrivial solutions, a result which suggests that it happens only in the trivial case. We do not have a general proof of this, but that the free-quark model is exceptional can be seen intuitively as follows. For any of the primitive solutions, $B=\frac{1}{2}(R+1)$ say, let Λ be the projection on the timelike states, and let the current $\exp(i\vec{k}\cdot\vec{X})$ commute[11] with Λ. Since M^2 commutes with Λ, this suggests that $[\vec{X}, M^2]$ and $[\vec{X},[\vec{X},M^2]]$ commute with Λ, and since

$$M^2=g_0-g_3+g_-(1-\epsilon)-1g_+,\quad [\vec{X},M^2]=2ig(1-\epsilon)^{-1},$$

$$[X_i,[X_j,M^2]]=-2\delta_{ij}\epsilon(1-\epsilon)^{-1},$$

this in turn suggests that $g=(\epsilon,\vec{g},g_0-g_3)$ commutes with Λ.[12] Let us assume this. Now from (9) and (10), Λ projects on $0\leq\epsilon<1$ and K_3 dilates ϵ ($M_{++}=0$ for primitive solutions). Hence $\Lambda(t)=\exp(itK_3)\Lambda\exp(-itK_3)$ is the spectral family[13] for ϵ. But $\exp(itK_3)$ maps g into itself. Hence g commutes with $\Lambda(t)$, and consequently with ϵ. From this it is easy to show that g is Abelian, and this is just the content of the free-quark model.[14]

If in general the spacelike solutions do indeed couple, the program of saturating with states of a single isospin fails—the restriction to the $I=\frac{1}{2}$ states is apparently too strong. Note that we have made no explicit use[15] of the discreteness of M^2, and hence can allow any amount of continuum so long as it has $I=\frac{1}{2}$. Thus, a partly continuous mass spectrum alone will not provide a cure. A continuum with increasing isospin (which is of course provided by the physical many particle states) seems to be necessary.

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[1]R. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966); M. Gell-Mann, in Proceedings of the International School of Physics "Ettore Majorana," Erice, Italy, 1966, edited by A. Zichichi (Academic Press, Inc., New York, 1966).

[2]Trivial solutions are $M=$const, in which case the equations of Table I can be solved exactly, and the two-free-quark model, for which the spectrum of M^2 is continuous.

[3]A nontrivial solution, which turns out to correspond to the special case of our primitive solution $B=0$ in which the operators $\vec{X}$ form a complete commuting set, was found by H. Leutwyler, Phys. Rev. Letters 20, 561 (1968).

[4]A nontrivial solution using a wave equation of the form (7), with a Dirac ⊗ Majorana representation of SL(2, C) was found by M. Gell-Mann, D. Horn, and J. Weyers, in Proceedings of the International Conference on Elementary Particles, Heidelberg, Germany, 1967, edited by H. Filthuth (North-Holland Publishing Company, Amsterdam, The Netherlands, 1968). This form of equation was first considered by E. Abers, I. Grodsky, and R. Norton, Phys. Rev. 159, 1222 (1967).

[5]In the case that one demands saturation of the algebra $[V_0(x), V_\mu(x')]$ for $\mu=1,2,3$, as well as $\mu=0$, the expansion definitely terminates, and there is some evidence to show that it terminates in the general case also. Hence the termination assumption is probably not too restrictive.

[6]In the case that one demands saturation of $[V_0(x), V_\mu(x')]$ with $\mu=0,1,2,3$, the only class of solutions is the primitive class $B=0$. In the limiting case of large g_μ and s, we have the simplified solution $B=R+1=0$, which can be derived from a linear wave equation $(2gp+s)\varphi=0$. A unified way of writing both solutions is

$$(ap^2+2gp+s)\varphi=0,\quad a=\text{scalar}.$$

A similar modification should also be made on the nonprimitive solutions.

[7]The operator M_{++} of Table I can be understood intuitively in terms of the wave Eqs. (6) and (7). It is nonzero if, and only if, $\mathfrak{M}$ contains a term $\gamma_\mu\gamma_\nu\Sigma_{\mu\nu}$, where $\Sigma_{\mu\nu}$ is a p-independent, γ-independent, SL(2, C) antisymmetric tensor.

[8]A sufficient technical condition is that there is a common, dense, invariant domain D for the operators of Table I, on which M^2 and ϵ are essentially self-adjoint, M_{++} and M_{--} are essentially adjoint, and J_3, $\vec{X}$, and K_3 generate a unitary group which leaves D invariant. The strongest assumption is that K_3 generates a unitary group. However, since for the wave equations (5)-(7) this is necessary in order to have finite Lorentz transformations in the spinor space, it is clearly a reasonable assumption. (Note that K_3 does not generate a unitary representation on the subspace of timelike solutions alone.)

[9]The above result concerning spacelike solutions is similar to a result obtained earlier by I. Grodsky and R. Streater, Phys. Rev. Letters 20, 695 (1968), but it is not completely equivalent. In our case, we assume current algebra and isospin factorization, but make no a priori assumptions concerning the representations of SL$(2,C)$ to which the particles belong or concerning the p independence of the current. [In fact, for the primitive solution $B=0$ and the general wave Eqs. (7) the current turns out to be linear in p.] Grodsky and Streater, on the other hand, do not assume current algebra, but assume that the current is p independent, and restrict the representations of SL$(2,C)$ by demanding a polynomial bound on the positive frequency projection operator. Finally, our result shows that the spacelike part does not completely decouple at $p_z \to \infty$. It is coupled by the operator $\exp(ixK_3)$, if not by the current.

[10]S. J. Chang and L. O'Raifeartaigh, Phys. Rev. 170, 1316 (1968), and Phys. Rev. (to be published).

[11]Note that the decoupling is characterized by the fact that whereas the full Lorentz goup in spinor space does not leave the timelike subspace invariant, the $E(2)$ subgroup generated by J_3 and $\vec{X}$ does.

[12]The argument up to this point is not rigorous because M^2 contains the singular term $(1-\epsilon)^{-1}$. Hence it does not follow immediately that $[\vec{X},M^2]$, $[\vec{X},[\vec{X},M^2]]$, and g_μ commute with Λ if M^2 and $\vec{X}$ do. Indeed, for the free-quark model itself, the anticommutator $[M^2,\vec{X}]_+$ does not commute with Λ. However, one expects the commutators to be better behaved than the anticommutators, especially as they can be derived from the (finite) expansion of $\exp(i\vec{k}\cdot\vec{X})M^2\exp(-i\vec{k}\cdot\vec{X})$ which, by hypothesis, commutes with Λ for all $\vec{k}$.

[13]We assume for simplicity that $\epsilon \geq 0$. One can check that there is no inconsistency, and practically no loss in generality, in making this assumption.

[14]In the general, nonprimitive case one has the somewhat looser argument: $M^2, J_\pm, J_3$ commute with Λ and K_3 does not. Hence if $\vec{X}$ commutes with Λ, the six basic variables of Table I, Sec. 2, commute with Λ, but K_3, which is a bilinear in these variables, does not. This appears to be a rather exceptional situation.

[15]Provided the formal transition from discrete to continuous masses is sufficiently smooth.