FURTHER RESULTS FOR THE MANY-BODY PROBLEM IN ONE DIMENSION

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The problem of N particles interacting in one dimension by a repulsive delta-function potential is solved, when the wave function ψ transforms like any irreducible representation R_ψ of S_N for which the Young tableau consists of a finite number of either rows or columns.

In a recent paper, Yang[1] gave a general formulation by which the one-dimensional repulsive delta-function interaction problem

$$H=-\sum_1^N \frac{\partial^2}{\partial x_i^{\,2}}+2c\sum_{i<j}\delta(x_i-x_j)\quad (c>0) \tag{1}$$

is reduced explicitly to a matrix problem. In particular, if we require the wave function ψ to transform like the irreducible representation R_ψ of the permutation group S_N, then ψ is given by Bethe's hypothesis:

$$\psi=[Q,P]\exp i[p_{P1}x_{Q1}+\cdots+p_{PN}x_{QN}], \tag{2}$$

if the p's satisfy a particular matrix equation

$$\exp(ip_jL)=\mu_j, \tag{3}$$

$$\mu_j\varphi=X_{j+1,j}'\cdots X_{N,j}'X_{1,j}'\cdots X_{j-1,j}'\varphi \quad (j=1,\cdots,N). \tag{4}$$

Here,

$$X_{ij}'=\frac{1+x_{ij}P_{ij}}{1+x_{ij}}, \tag{5}$$

$$x_{ij}=\frac{ic}{p_i-p_j}. \tag{6}$$

P_{ij} are permutations in the representation $\tilde{R}_\psi=[m_1,m_2\cdots,m_\kappa]$.

Yang was able to solve this matrix equation for $\tilde{R}_\psi=[N-M,M]$ by taking as a representation of P_{ij} permutations of M identical particles and $N-M$ identical vacancies, on a cyclic chain. He then showed the wave function of the particles, in this new problem, to be given by a generalized Bethe's hypothesis, which we shall call the Bethe-Yang hypothesis.

The success of this idea suggests that, for the general representation $\tilde{R}_\psi$, we represent P_{ij} by permutations of m_1 identical vacancies and $N-m_1$ distinguishable particles. One would then expect the wave function to satisfy the Bethe-Yang hypothesis for each ordering of the $N-m_1$ particles. This parallels the original δ-interaction problem. In this way, one might obtain a new matrix equation of smaller dimension. It is the purpose of this note to demonstrate that this scheme is, in fact, possible. We shall carry through the program for $\kappa=3$, $\tilde{R}_\psi=[N-M,M-M_1,M_1]$; the generalization to arbitrary κ will be evident in the end.

The P_{ij} of Eq. (4) are permutation operators of $\tilde{R}_\psi$ of S_N. We may represent them by M distinguishable particles on a cyclic chain of N sites. In addition, rearrangements of the M particles constitute a representation of S_M.

If $1\leq y_{Q1}<\cdots<y_{QM}\leq N$ are the coordinates of the M particles, and $\Lambda_1,\cdots,\Lambda_M$ a set of unequal real numbers, then we seek a solution to Eq. (4) of the form

$$\varphi=\sum_P[Q,P]F(\Lambda_{P1},y_{Q1})\cdots F(\Lambda_{PM},y_{QM}), \tag{7}$$

$$F(\Lambda,y)=\prod_{j=1}^{y-1}\frac{ip_j-i\Lambda-c'}{ip_{j+1}-i\Lambda+c'}\quad (c'=c/2). \tag{8}$$

P,Q are two permutations of the integers $1,\cdots,M$. We arrange $[Q,P]$ as an $M!\times M!$ matrix, and denote the columns by ξ_P.

One finds Eq. (4) to be satisfied, except for periodicity, if

$$\xi_{\cdots\alpha\beta\cdots}=Y_{\beta\alpha}'^{34}\xi_{\cdots\beta\alpha\cdots}, \tag{9}$$

where

$$Y_{\alpha\beta}'^{34}=\frac{x_{\alpha\beta}+P_{34}}{1-x_{\alpha\beta}}, \tag{10}$$

$$x_{\alpha\beta}=\frac{ic}{\Lambda_\alpha-\Lambda_\beta}.$$

It is to be understood that the $P_{\alpha\beta}$ of Eq. (10) are now the permutation operators of an as yet unspecified representation R_φ of S_M. Since these Y' operators satisfy the same identities as (Y7), (Y8), Eqs. (9) are consistent. Imposing periodic boundary conditions, we obtain the equations

$$\nu_\alpha' = \prod_{j=1}^{N}\left(\frac{i\Lambda_\alpha - ip_j + c'}{i\Lambda_\alpha - ip_j - c'}\right), \tag{12}$$

$$\nu_\alpha'\zeta_0 = X_{\alpha+1,\alpha}''\cdots X_{M,\alpha}''X_{1,\alpha}''\cdots X_{\alpha-1,\alpha}''\zeta_0 \quad (\alpha = 1,\cdots,M), \tag{13}$$

where

$$X_{\alpha\beta}'' = P_{\alpha\beta}Y_{\alpha\beta}'^{\alpha\beta} = \frac{1+x_{\alpha\beta}P_{\alpha\beta}}{1-x_{\alpha\beta}} = \frac{1+x_{\alpha\beta}}{1-x_{\alpha\beta}}X_{\alpha\beta}'. \tag{14}$$

Here $X_{\alpha\beta}'$ is X_{ij}' with $x_{\alpha\beta}$ replacing x_{ij}. We may thus rewrite Eqs. (12) and (13) as

$$\nu_\alpha' = -\nu_\alpha \prod_{\beta=1}^{M}\left(\frac{i\Lambda_\alpha - i\Lambda_\beta + c}{i\Lambda_\alpha - i\Lambda_\beta - c}\right) = \prod_{j=1}^{N}\left(\frac{i\Lambda_\alpha - ip_j + c'}{i\Lambda_\alpha - ip_j - c'}\right), \tag{12'}$$

$$\nu_\alpha\zeta_0 = X_{\alpha+1,\alpha}'\cdots X_{M,\alpha}'X_{1,\alpha}'\cdots X_{\alpha-1,\alpha}'\zeta_0 \quad (\alpha = 1,\cdots,M). \tag{13'}$$

We choose the representation R_φ and S_M to be $[m_2, m_3, \cdots, m_\kappa]$. Thus Eq. (4) is satisfied, and we find the eigenvalue to be

$$\mu_j = \prod_{\beta=1}^{M}\left(\frac{ip_j - i\Lambda_\beta - c'}{ip_j - i\Lambda_\beta + c'}\right). \tag{14'}$$

Although the problem at this point has only been reduced to another matrix equation (13′), we notice that this new equation is identical in form to the original equation (4). Thus we may use the same trick to again reduce R_φ. For κ a finite number of rows, this process will terminate after producing κ coupled algebraic equations.

For our case where $\kappa = 3$, we carry the procedure through once more, and obtain the equations

$$\exp(ip_jL) = \prod_{\beta=1}^{M}\left(\frac{ip_j - i\Lambda_\beta - c'}{ip_j - i\Lambda_\beta + c'}\right) \quad (j = 1,\cdots,N); \tag{15}$$

$$\prod_{j=1}^{N}\left(\frac{i\Lambda_\alpha - ip_j + c'}{i\Lambda_\alpha - ip_j - c'}\right) = -\prod_{\beta=1}^{M}\left(\frac{i\Lambda_\alpha - i\Lambda_\beta + c}{i\Lambda_\alpha - i\Lambda_\beta - c}\right)\prod_{b=1}^{M_1}\left(\frac{i\Lambda_\alpha - ik_b - c'}{i\Lambda_\alpha - ik_b + c'}\right) \quad (\alpha = 1,\cdots,M); \tag{16}$$

$$\prod_{\beta=1}^{M}\left(\frac{ik_a - i\Lambda_\beta + c'}{ik_a - i\Lambda_\beta - c'}\right) = -\prod_{b=1}^{M_1}\left(\frac{ik_a - ik_b + c}{ik_a - ik_b - c}\right) \quad (a = 1,\cdots,M_1). \tag{17}$$

We take the logarithm of Eqs. (15), (16), and (17):

$$L_p = 2\pi I_p + \sum_\Lambda \theta(2p - 2\Lambda), \tag{15'}$$

$$\sum_{\Lambda'}\theta(\Lambda - \Lambda') = 2\pi J_\Lambda + \sum_p \theta(2\Lambda - 2p) + \sum_k \theta(2\Lambda - 2k), \tag{16'}$$

$$\sum_{k'}\theta(k - k') = 2\pi K_k + \sum_\Lambda \theta(2k - 2\Lambda), \tag{17'}$$

where $\theta(x) = -2\tan^{-1}(x/c)$, and p, Λ, k are sets of N, M, M_1 ascending real numbers; I_p, J_Λ, K_k are either integers or half integers, coming from the logarithm. For the ground state, when M is even and N, M_1 are odd, we find

I_p = successive integers from $-\frac{1}{2}(N-1)$ to $+\frac{1}{2}(N-1)$, (18a)

$\frac{1}{2} + J_\Lambda$ = successive integers from $1 - M/2$ to $+M/2$, (18b)

K_k = successive integers from $-\frac{1}{2}(M_1-1)$ to $+\frac{1}{2}(M_1-1)$. (18c)

We may now approach the limit $N, M, M_1, L \to \infty$ proportionally, obtaining integral equations. After differentiating, these equations for the appropriate densities are

$$2\pi\rho = 1 + \int_{-B}^{B} \frac{4c\sigma d\Lambda}{c^2+4(p-\Lambda)^2}, \tag{19}$$

$$\int_{-Q}^{Q} \frac{4c\rho dp}{c^2+4(\Lambda-p)^2} + \int_{-R}^{R} \frac{4c\tau dk}{c^2+4(\Lambda-k)^2} = 2\pi\sigma + \int_{-B}^{B} \frac{2c\sigma d\Lambda'}{c^2+(\Lambda-\Lambda')^2}, \tag{20}$$

$$\int_{-B}^{B} \frac{4c\sigma d\Lambda}{c^2+4(k-\Lambda)^2} = 2\pi\tau + \int_{-R}^{R} \frac{2c\tau dk'}{c^2+(k-k')^2}. \tag{21}$$

In addition,

$$\frac{N}{L} = \int_{-Q}^{Q} \rho dp, \quad \frac{M}{L} = \int_{-B}^{B} \sigma d\Lambda,$$

$$\frac{M_1}{L} = \int_{-R}^{R} \tau dk; \tag{22}$$

the ground state energy is

$$\frac{E}{L} = \int_{-Q}^{Q} p^2 \rho dp. \tag{23}$$

For arbitrary finite κ, define a set of κ variables, densities, and limits, denoted by k_i, $\rho_i(k_i), B_i$. Let

$$M_i = \sum_{j=i}^{M} m_j.$$

Then the general equations are

$$2\pi\rho_1 = 1 + \int_{-B_2}^{B_2} \frac{4c\rho_2 dk_2}{c^2+4(k_1-k_2)^2}, \tag{24}$$

$$\int_{-B_{i+1}}^{B_{i+1}} \frac{4c\rho_{i+1}dk_{i+1}}{c^2+4(k_i-k_{i+1})^2} + \int_{-B_{i-1}}^{B_{i-1}} \frac{4c\rho_{i-1}dk_{i-1}}{c^2+4(k_i-k_{i-1})^2} = 2\pi\rho_i + \int_{-B_i}^{B_i} \frac{2c\rho_i dk_i'}{c^2+(k_i-k_i')^2} \quad (i=2,\cdots,\kappa-1), \tag{25}$$

$$\int_{-B_{\kappa-1}}^{B_{\kappa-1}} \frac{4c\rho_{\kappa-1}dk_{\kappa-1}}{c^2+4(k_\kappa-k_{\kappa-1})^2} = 2\pi\rho_\kappa + \int_{-B_\kappa}^{B_\kappa} \frac{2c\rho_\kappa dk_\kappa'}{c^2+(k_\kappa-k_\kappa')^2}, \tag{26}$$

$$\frac{M_i}{L} = \int_{-B_i}^{B_i} \rho_i dk_i \quad (i=1,\cdots,\kappa); \quad \frac{E}{L} = \int_{-B_1}^{B_1} \rho_1 k_1^2 dk_1. \tag{27}$$

We may also solve for a representation $R_\psi' = \tilde{R}_\psi$ by making use of the identity (Y13). The m_i are now the lengths of the κ columns. All equations remain the same, except Eq. (24), which becomes

$$2\pi\rho_1 + \int_{-B_2}^{B_2} \frac{4c\rho_2 dk_2}{c^2+4(k_1-k_2)^2} = 1 + \int_{-B_1}^{B_1} \frac{2c\rho_1 dk_1'}{c^2+(k_1-k_1')^2}. \tag{24'}$$

These sets of integral equations are generalized Fredholm equations with a symmetric nonsingular kernel. If some $B_i = \infty$, $i \neq 1$, we integrate the ith equation over all k_i, giving $m_{i+1} = m_i$. Thus all $B_i = \infty$, $i \neq 1$, corresponds to a rectangular tableau.

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[1]C. N. Yang, Phys. Rev. Letters 19, 1312 (1967). References to particular equations of this paper will be designated by Y plus the number of the equation; thus, Y1 is Eq. (1) of the paper. This paper gives earlier references to work on the problem.