KRONECKER DELTAS IN ANGULAR MOMENTUM FOR WEAK PROCESSES

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Kronecker deltas in angular momentum for weak processes are shown to be required by current algebra and conspiracy relations. Theoretical and experimental consequences of such nonanalytic pieces are discussed.

It has recently been observed that certain weak amplitudes are forced by the nonvanishing of equal-time commutation relations to have fixed poles in the complex angular-momentum plane.[1] Such fixed poles, at nonsense points[2] of both right and wrong signature, are in general allowed in weak processes (at least in lowest order in the weak coupling) since linear unitarity does not provide a mechanism for removing them. In strong processes fixed poles can presumably appear, and in general do,[3] at nonsense wrong-signature points. The purpose of this note is to show that fixed poles in partial-wave helicity-flip amplitudes at nonsense points will in general be accompanied by Kronecker deltas in partial-wave helicity-nonflip amplitudes at sense points.[4]

We consider the covariant scattering amplitude $T_{\mu\nu}(\nu,t,q^2)$, for the elastic process

$$\gamma_{q_1,\nu}+\pi_{p_1}\rightarrow\gamma_{q_2,\mu}+\pi_{p_2}, \tag{1}$$

where $\gamma_{q_1,\nu}$ represents an isovector photon of momentum q_1 and polarization index ν, and π_{p_1} represents a pion of momentum p_1. We have $P=\frac{1}{2}(p_1+p_2)$, $\nu=q_1\cdot P$, $t=(p_1-p_2)^2$, $q_1^2=q_2^2=q^2$, $p_1^2=p_2^2=m^2$, and we restrict ourselves to $I=1$ in the t channel. The invariant amplitude A_1, which appears in the expansion

$$T_{\mu\nu}=P_\mu P_\nu A_1+\cdots, \tag{2}$$

behaves for large energies (for $t<m_\rho^2$) like

$$A_1(\nu,t,q^2)\underset{\nu\to\infty}{\longrightarrow}F_\pi(t)/\nu, \tag{3}$$

where $F_\pi(t)$ is the coefficient of P_μ in the form factor $\langle\pi_{p_2}|V_\mu(0)|\pi_{p_1}\rangle$. This is the content of the Fubini–Dashen–Gell-Mann sum rule.[5] The t-channel helicity-flip–2 amplitude is simply given by

$$F_{1-1;00}{}^t(\theta_t,t)=2\left(\frac{t-4m^2}{4}\right)\sin^2\theta_t A_1(\nu,t,q^2)\underset{\nu\to\infty}{\longrightarrow}\left(\frac{8\nu}{t-4q^2}\right)F_\pi(t). \tag{4}$$

This implies that the analytically continued partial-wave amplitude, $f_{1-1;00}{}^J(t)$, has a fixed pole at the sense-nonsense point $J=1$, with residue proportional to $F_\pi(t)$.[1]

The other helicity amplitudes also get contributions from A_1. If we neglect the other invariant amplitudes (or assume for them pure Regge behavior), then we would have at large ν $(t<m_\rho^2)$

$$F_{11;00}{}^t(\nu,t,q^2)\underset{\nu\to\infty}{\longrightarrow}-2\left(\frac{t-4m_\pi^2}{4}\right)\sin^2\theta_t A_1\approx-\frac{8\nu}{t-4q^2}F_\pi(t). \tag{5}$$

The amplitudes $F_{00;00}{}^t$ and $F_{10;00}{}^t$ get no contribution from A_1 once gauge invariance is used. These asymptotic pieces correspond to Kronecker deltas at the sense right-signature point $J=1$ (one cannot, of course, have a fixed pole at a sense value). It could be the case that other invariant amplitudes have non-Regge pieces in them, and conceivably these could conspire to cancel in the helicity-nonflip amplitudes. However we shall show that this is impossible.

To see this we utilize the conspiracy relation, which follows from the vanishing of the s-channel helicity-flip amplitudes in the forward direction ($t=0$). Expressed in terms of t-channel helicity amplitudes,[6] we have

$$F_{00;00}^{t}(z_t, t=0)-F_{11;00}^{t}(z_t, t=0)=F_{1-1;00}^{t}(z_t, t=0). \tag{6}$$

The important feature of this relation is that it couples a spin-flip amplitude to spin-nonflip amplitudes. Therefore, unless we have evasion, or the vanishing of the residue of the fixed pole at $t=0$ [in the case of vector currents $F_\pi(0)=1$], a fixed pole at $J=1$ in $F_{1-1;00}^{t}$ necessarily implies a Kronecker delta in $F_{00;00}^{t}-F_{11;00}^{t}$.

To exhibit explicitly this nonanalytic piece, we consider the conspiracy relation as expressed for the partial-wave (odd-signatured) helicity amplitudes. For $J>N$ [where N is the number of subtractions necessary in fixed-t dispersion relations for $F^t(\nu,t)$], (6) implies

$$h_{00}^{J}(t=0)-h_{00}^{J+2}(t=0)=(J+3)(J+4)h_{1-1}^{J+2}(t=0)-(J-1)Jh_{1-1}^{J}(t=0), \tag{7}$$

where

$$F_{1-1;00}^{t}(z_t,t)=\sum_{J=3,5}^{\infty}(2J+1)[(J-1)J(J+1)(J+2)]^{\frac{1}{2}}d_{20}^{J}(z_t)h_{1-1}^{J}(t),$$

$$F_{00;00}^{t}(z_t,t)-F_{11;00}^{t}(z_t,t)=\sum_{J=1,3}^{\infty}(2J+1)d_{00}^{J}(z_t)h_{00}^{J}(t), \tag{8}$$

and h_{1-1}^{J}, h_{00}^{J} are analytic in J for $J>N$.

However, if we project the partial-wave amplitudes from (6) for integer J, we derive

$$h_{00}^{1}(t=0)-h_{00}^{3}(t=0)=20h_{1-1}^{3}(t=0). \tag{9}$$

This does not coincide with the analytic continuation of (7), from large J down to $J=1$, if $h_{1-1}^{J}(t=0)$ has a fixed pole at $J=1$. Thus

$$h_{00}^{J}(t=0)=\tilde{h}_{00}^{J}(t=0) + \delta_{J,1}\lim_{J\to 1}(J-1)h_{1-1}^{J}(t=0), \tag{10}$$

where $\tilde{h}_{00}^{J}$ is analytic in J.[7]

From the conspiracy relation alone, it is impossible to say whether these subtraction terms will appear in F_{00}^{t} or in F_{11}^{t} or in both. If indeed A_1 is the only invariant with non-Regge asymptotic behavior, then the subtraction terms (the coefficients of the Kronecker deltas) will be given by (5).[8] However, in models such as that given in Ref. 1, other invariant amplitudes will have non-Regge pieces. Also these weak amplitudes can have fixed poles at the other nonsense points (in this case $J=0$), even though current algebra does not inform us of their existence.[9] Therefore, our conclusion is that weak amplitudes will in general have Kronecker deltas, related to fixed poles, at nonsense points of angular momentum, and without further dynamical information the coefficients of the Kronecker deltas will be undetermined.

In consequence, the helicity amplitudes of weak processes do not, in general, have the same asymptotic behavior as the amplitudes of strongly interacting particles; and additional subtractions will be necessary in writing dispersion relations for them. To support the opposite point of view, Dashen and Frautschi[10] have argued that if these weak amplitudes fall off sufficiently fast as the "masses" of the currents go to infinity, one can write unsubtracted dispersion relations in the masses and relate the asymptotic behavior to that of strong amplitudes, thereby deriving Regge behavior for the weak process.[10] However, current algebra informs us that there are pieces, such as the residue of the fixed pole at $J=1$, that do not vanish for large masses of the currents, and hence one cannot assume unsubtracted dispersion relations in the external mass. In fact, the subtraction terms in ν in weak amplitudes will also be subtraction terms in the masses of the external currents. Needless to say, the Dashen-Frautschi program of bootstrapping currents and deriving commutation relations requires re-evaluation.

The nonanalytic pieces in the partial-wave amplitudes contribute in a definite way to the asymptotic behavior of weak processes and hence are experimentally detectable. Qualitatively, however, they will contribute in the same way as do fixed poles to the high-energy behavior; and only detailed polarization experiments, which isolate helicity-nonflip crossed-channel amplitudes, can distinguish Kronecker deltas from fixed poles. Such contributions can be expected to appear in the differential cross sections for the processes $\gamma+p\to\gamma+p$, for which it has been argued that there is a fixed pole at $J=0$, $I=1$,[9] and $\gamma+p\to n+e^++\nu$ for which current algebra requires a fixed pole at $J=1$, $I=1$ (and hence Kronecker deltas).

In the case of strong-interaction amplitudes, such as $\pi\rho$ scattering, the same arguments would lead one to the conclusion that, unless the residue of the fixed poles at nonsense wrong-signatured values of J vanish at $t=0$,[11] the partial-wave helicity-nonflip amplitude at a sense wrong-signature point is nonanalytic in J. The presence of such Kronecker deltas in the signatured amplitude is of no physical consequence, since the signatured amplitude is defined in terms of the physical amplitude only up to a polynomial of the wrong signature. Such a polynomial is manifested in the partial-wave amplitudes by Kronecker deltas but has no influence on our ability to utilize the t-channel unitarity conditions on the partial-wave signatured amplitude, $\mathrm{Im}A^{(+)}(J,t)=\lambda|A^{(+)}(J,t)|^2$. If the amplitude has a nonanalytic piece at $J=1$, $A^{(+)}(J,t)=\tilde{A}^{(+)}(J,t)+\delta_{J1}F(t)$ where $\tilde{A}^{(+)}(J,t)$ is analytic in J, it follows that $\mathrm{Im}\tilde{A}^{(+)}(J,t)=\lambda|\tilde{A}^{(+)}(J,t)|^2$, and $\mathrm{Im}F(t)=\lambda|F(t)|^2+\lambda[F^*(t)\tilde{A}(1,t)+\tilde{A}^*(1,t)F(t)]$, so the analytic piece will satisfy elastic unitarity by itself.

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[4]The existence of Kronecker deltas in special perturbative models has been shown by S. Mandelstam, Nuovo Cimento 30, 1148 (1963).

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[7]One should not confuse this nonanalytic piece with that arising from the exchange of an elementary particle in the t channel. Even if the coefficient of the Kronecker delta at $J=1$ has a pole at $t=m_\rho^2$, this does not mean that the ρ meson is elementary. Such a conclusion would only be reached if the analytic (in J) part of the partial-wave amplitude does not have a Regge pole corresponding to the ρ.

[8]In this case, although F^t will have a subtraction in ν (for fixed t), it would be determined analytically through the Fubini–Dashen–Gell-Mann sum rule.

[9]In fact, we have given elsewhere (D. J. Gross and H. Pagels, to be published) phenomenological reasons for the existence of a fixed pole at $J=0$ in the amplitude for virtual Compton scattering. It is possible that the residues of fixed poles at the points in the J plane beneath the highest nonsense point are related to the equal-time commutators of time derivatives of the currents with themselves.

[10]R. F. Dashen and S. C. Frautschi, Phys. Rev. 145, 1287 (1966). An implicit assumption made by these authors is that one has uniform convergence in ν for the (unsubtracted) dispersion relations in the masses. If this is false, then one could recover the non-Regge behavior of the weak amplitudes from unsubtracted dispersion relations in the masses, since the interchange of the integration over the masses and the high-energy limit would no longer be possible.

[11]If this happens, then one can derive superconvergence relations of the type discussed by J. Schwartz, Phys. Rev. 159, 1269 (1967). For the particular case of $\rho\pi$ scattering, simple saturation of this sum rule indicates that it is unlikely that the residue vanishes at $t=0$.