REFORMULATION OF THE DIRAC THEORY OF THE ELECTRON

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It is shown that when the Dirac theory is formulated in the formalism of the dynamical group O(4, 2), there is no need to invoke the hole theory or the notion of backward motion in time to describe antiparticles. A single irreducible representation of O(4, 2) is used and the discrete operators P, C become inner automorphisms of the group. The most general linear minimal parity-violating equation generalizing the Dirac equation is shown to lead to two possible distinct mass values.

It is clear that the Dirac equation goes beyond the customary definition of an elementary particle as an irreducible representation of the Poincaré group: One has to require in addition the existence of a vector operator Γ_μ in the representation space which only happens if one takes the direct sum of at least two finite-dimensional representations of the Lorentz group.[1] In the usual interpretation of the Dirac equation, one has then to deal with the negative energy states, invoking the standard prescriptions of the hole theory,[2] or the notion of particles moving backward in time.[3]

However, if we go to the larger group O(4, 2), containing the Lorentz group O(3, 1), the particle-antiparticle system is described in one irreducible representation, the vector operator Γ_μ exists in the representation space, and the sign of the energy becomes a new internal quantum number in the rest frame. Thus both particles and antiparticles can be boosted to positive energies. Current conservation then implies equal mass for both particles and antiparticles. Discrete operations of parity and charge conjugation become now simple inner automorphisms of the group whereas they are outside of the Lorentz group. Thus a single-particle theory of the electron-positron system is possible, the general opinion at present being that a relativistic theory must necessarily imply an infinite number of particles.

The formalism that we use exactly parallels recent work on the O(4, 2) theory of electromagnetic interactions and the H atom,[4] and of hadron properties.[5] It is remarkable that the same dynamical group O(4, 2) can be used to describe the properties and interactions of such widely different structures as leptons, the H atom, and hadrons. Only the Casimir invariants of the group distinguish these different structures.

The fundamental irreducible four-dimensional representation of the O(4, 2) algebra is[6] given by the following: $L_{ij}=\frac{1}{2}i\gamma_i\gamma_j$ (spin), $L_{i4}=\frac{1}{2}i\gamma_5\gamma_i$ (the analog of Lenz vector), $L_{i5}=M_i=\frac{1}{2}i\gamma_i\gamma_0$ (pure Lorentz transformations), $\Gamma_\mu=\frac{1}{2}\gamma_\mu$ (algebraic current operator), $L_{45}=-i\frac{1}{2}\gamma_5\gamma_0$ (the "tilt"), and $L_{46}=-\frac{1}{2}\gamma_5$. We use $\gamma_a\gamma_b+\gamma_b\gamma_a=2g_{ab}$, the metric $(----++)$, $\gamma_5=\gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_i^2=\gamma_5^2=-1$, $\gamma_0^2=+1$.

Quite generally, the rest-frame states in O(4, 2) are labeled by three quantum numbers and parity, $|njm\pm\rangle$, where n is the eigenvalue of L_{56}, $j(j+1)$ the eigenvalue of L^2, and m that of L_{12}. In the above fundamental representation, even L_{34} can be diagonalized together with L_{56}, L^2, and L_{12}. In terms of these quantum numbers we have the four basis states

$$|\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},+\rangle,\quad |\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},+\rangle,\quad |-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},-\rangle,\ \text{and}\ |-\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2},-\rangle, \tag{1}$$

where the parity operator for this representation is defined to be $P=\gamma_0$. The first Casimir operator has the value $Q=L_{ab}L^{ab}=15/4$. The parabolic quantum numbers n_1, n_2, i.e., the eigenvalues of $\frac{1}{2}(L_{56}\pm L_{34})$, give in this case, unlike the infinite-dimensional case, the same states: $|njm\pm\rangle\equiv|n_1n_2m\pm\rangle$, $n=n_1+n_2$.

Charge conjugation is another inner automorphism,

$$C'=i\begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix},$$

and clearly connects different eigenvalues of L_{56} given in (1).

Next, we define, as in the previous work on O(4, 2),[4-7] spinorial wave functions for states with momentum $p^\mu=(m\cosh\xi, \hat{\xi}m\sinh\xi)$ by

$$|njm\pm,p\rangle=e^{i\vec{\xi}\cdot\vec{\mathrm{M}}}|njm\pm\rangle = \left\{\cosh\tfrac{1}{2}\xi+\hat{\xi}\cdot\begin{pmatrix}0 & \vec{\sigma}\\ \vec{\sigma} & 0\end{pmatrix}\sinh\tfrac{1}{2}\xi\right\}|njm\pm\rangle. \tag{2}$$

The scalar vertex function connecting any two states is given by

$$S=g_S\langle n''j''m'',p''|n'j'm',p'\rangle=g\mathfrak{V}_{n''n'}(\vec{\xi}),\quad \vec{\xi}=\vec{\xi}'-\vec{\xi}'', \tag{3}$$

and is the analog of $\bar{u}(p'')u(p)$ in the usual formalism. The functions $\mathfrak{V}_{n'n}$ are the basic quantities entering into all higher order calculations and are given by [in the basis (1)][8]

$$\mathfrak{V}_{n'n}=\begin{bmatrix} \cosh\frac{1}{2}\xi & 0 & -\xi_3\sinh\frac{1}{2}\xi & -(\xi_1+i\xi_2)\sinh\frac{1}{2}\xi \\ 0 & \cosh\frac{1}{2}\xi & -(\xi_1-i\xi_2)\sinh\frac{1}{2}\xi & \xi_3\sinh\frac{1}{2}\xi \\ \xi_3\sinh\frac{1}{2}\xi & (\xi_1+i\xi_2)\sinh\frac{1}{2}\xi & -\cosh\frac{1}{2}\xi & 0 \\ (\xi_1-i\xi_2)\sinh\frac{1}{2}\xi & -\xi_3\sinh\frac{1}{2}\xi & 0 & -\cosh\frac{1}{2}\xi \end{bmatrix}. \tag{4}$$

They are normalized at $\vec{\xi}=0$ to[9]

$$\mathfrak{V}_{n'n}(0)\equiv\langle n'j'm'|njm\rangle=(2n')\delta_{n'n}. \tag{5}$$

The vector vertex function is defined by

$$\mathfrak{F}_\mu=g_V\langle n'j'm',p'|j_\mu|njm,p\rangle. \tag{6}$$

We require from $\mathfrak{F}_\mu$ that the charge of each state is correctly given and the equation of current conservation is

$$\langle n|j_0|n\rangle=e_n, \tag{7}$$

$$(p'-p)^\mu j_\mu=0, \tag{8}$$

which becomes[7]

$$m_{n'}\langle n'|j_0|n,p\rangle-m_n\langle n',-p|j_0|n\rangle=0. \tag{9}$$

The most general parity-conserving current operator can be written as

$$j_\mu=e\gamma_\mu+i(e\kappa/2m)\sigma_{\mu\nu}q^\nu. \tag{10}$$

The second term does not contribute to the charge and is always conserved. In order to satisfy the requirement (7) for the first term, we must take e to be positive for the positive-n states and negative for the negative-n states. Note that in the theory of infinite multiplets, like the H atom, the coefficient e is in general a matrix.[4,5] We have then from the second requirement (9) and (5)

$$\{m_{n'}(2n')^2-m_n(2n)^2\}\langle n'|np\rangle=0,$$

or

$$m_{n'}=m_n, \tag{11}$$

for all states. Conversely, the requirement of positive masses for all states implies that the negative-n states must have negative charge, hence the existence of antiparticles. There is another equivalent procedure to the use of the charge matrix e followed in the infinite multiplet theory[4,5,7]: Take in Eq. (10), $e=1$, but let the current j_μ act on "physical states" $|\bar{n}\rangle=\mathfrak{N}^{-1}|n\rangle$ and determine $\mathfrak{N}$ from Eq. (7), i.e., $\langle\bar{n}|j_0|\bar{n}\rangle=e_n$.

With these results the scattering and annihilation vertices are given by

$$\langle n|j_\mu|np\rangle=\textstyle\sum_{n''}(j_\mu)_{nn''}\mathfrak{V}_{n''n}(\xi),$$

$$\langle n|j_\mu|n',-p\rangle=\textstyle\sum_{n''}(j_\mu)_{nn''}\mathfrak{V}_{n''n'}(-\xi), \tag{12}$$

respectively, and we recover the usual rules of covariant perturbation theory and of crossing symmetry.[10]

So far we have not written down the Dirac equation as such, but the above formalism and the requirement of current conservation is equivalent to the Dirac equation. Where we go beyond the Dirac equation is in tne use of the irreducible representation of O(4, 2) and thereby the introduction of the new quantum number n.[11]

The most general minimal linear conserved current in O(4, 2) is[5]

$$j_\mu=\alpha_1\Gamma_\mu+\alpha_2P_\mu+\alpha_3P_\mu L_{46}. \tag{13}$$

In the present case the second term is a linear combination of γ_μ and $\sigma_{\mu\nu}q^\nu$. Hence we are left with two terms $\alpha_1\Gamma_\mu+\alpha_3P_\mu L_{46}$. In particular, the current operator of the H atom has exactly these two terms. The term $\alpha_3P_\mu L_{46}$ in the present four-dimensional representation is $-P_\mu\gamma_5$ and does not conserve parity. Therefore the Dirac equation is the most general four-component parity-conserving equation with a minimal current. However, in situations where parity is not conserved, the most general equation is

$$(P^\mu\alpha_1\gamma_\mu-\alpha_3m^2\gamma_5+\beta\gamma_5-\gamma)|\bar{n},p\rangle=0. \tag{14}$$

We note that the equation of the H atom is exactly of this form, where, however, L_{46} is parity conserving. In (14), $|\bar{n},p\rangle$ are the so-called "tilted" states also occurring in H-atom and hadron calculations (for covariance, the tilting operation is always done before boosting):

$$|\bar{n},p=0\rangle = e^{\frac{1}{2}\theta\gamma_5\gamma_0}|n,p=0\rangle. \qquad (15)$$

In such a theory the vertex function is given by

$$\langle\bar{n}'|j_\mu|\bar{n}p\rangle = \langle n'|e^{-\frac{1}{2}\theta\gamma_5\gamma_0}(\alpha_1\gamma_\mu - \alpha_3 P_\mu\gamma_5) \times e^{i\vec{\xi}\cdot\vec{M}}e^{\frac{1}{2}\theta\gamma_5\gamma_0}|n\rangle.$$

The mass spectrum can be obtained either from Eq. (14), or from the current conservation requirement (9), first by operating with $e^{-i\vec{\xi}\cdot\vec{M}}$ and then with the inverse of (15), $e^{-\frac{1}{2}\theta\gamma_5\gamma_0}$. It is interesting that one gets now two mass values; for example, for $\beta=0$ in (14) one finds

$$m^2 = (2\alpha_3^2)^{-1}[\alpha_1^2 \pm (\alpha_1^4 - 4\gamma^2\alpha_3^2)^{1/2}]. \qquad (16)$$

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[8]Actually only the diagonal elements of $\mathcal{V}_{nn'}$ are physical because of the conservation of total charge in the S matrix. This can be taken into account by requiring the invariance of the S matrix under the compact subgroup O(2), rotation in the 56 plane, analog to the compact subgroup O(3) for angular momentum.

[9]Note that Eq. (5) is not the Hilbert-space norm of the states, but the scalar $\bar{u}(0)u(0)$. Because the representation in spin space is not unitary, $D^{-1} \neq D^\dagger$; hence these two things are different.

[10]See Ref. 8 for charge conservation.

[11]If we write an equivalent Dirac equation, it will have the form $[\eta\gamma_\mu p^\mu - M_0]\psi = 0$, i.e., the charged quantum number occurring in the free-particle equation. This is appropriate because it is an internal quantum number of the free system in the present interpretation.

AN EVALUATION OF SEARCHES FOR C NONCONSERVATION IN ETA DECAY

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Since the suggestion was made[1] that C invariance might not hold for the electromagnetic interaction, a number of attempts[2-13] have been made to find evidence for a C nonconservation in the electromagnetic decay of the eta meson. To date, there is no experimental evidence for the existence of the C-nonconserving decay $\eta \to \pi^0 e^+e^-$ (Table I). Furthermore, although measurements of the asymmetry in the Dalitz plot $\eta \to \pi^+\pi^-\pi^0$ show disagreement (Table II), the most precise of these[9] gives a null result. Similarly, the decay $\eta \to \pi^+\pi^-\gamma$ shows no charge asymmetry.[12] We would like to consider whether these experimental results are compatible with an electromagnetic C nonconservation of strength sufficient to account for the observed CP nonconservation in K_2^0 decay.[14] In doing so, we take special note of the following considerations:

(1) The width for the decay $\eta \to \gamma\gamma$ as measured experimentally[15] is an order of magnitude larger than earlier theoretical estimates.[16]

(2) The decay $\eta \to \pi^0 e^+e^-$ and the asymmetry in