generates the form factor as an eigenfunction of the Saxon-Woods potential U_{WD}, fails to give an accurate form factor for the $f_{5/2}$ case.

Figure 2 shows the result of the calculation of the pseudopotential, $U_{3/2}^{P}(r)$, for the $p_{3/2}$ form factor. The potential $U_j^{C}(r)$, for $j=\frac{3}{2}$, is also shown. On performing the conventional "well-depth" calculation of the form factor, it is found that U_{WD}, for this case, has a depth $V_0=-56$ MeV. From Fig. 2 it is seen that, aside from a "pole,"[11] $U_{3/2}^{P}(r)$ starts with a depth of about 3 MeV near the origin and tapers off as the radial distance increases. Since $U_{3/2}^{C}(r)$ is a Saxon-Woods potential of depth 53.3 MeV, the addition of $U_{3/2}^{P}(r)$ to $U_{3/2}^{C}(r)$ would give roughly a Saxon-Woods potential of depth ≈56.3 MeV. It is, therefore, understandable why the conventional "well-depth" prescription succeeds in giving a good form factor for the $p_{3/2}$ case.

The calculation of $U_{1/2}^{P}(r)$, for the $p_{1/2}$ form factor, gave results similar to those discussed for the $p_{3/2}$ case.

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EXPLANATION OF THE MASS ASYMMETRY IN NUCLEAR FISSION*

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The asymmetric mass yield distribution in spontaneous fission is simply related to the intrinsic nuclear structure and to the nuclear spin-orbit coupling.

The phenomenon of nuclear fission is the only well-known one in which a system of many strongly interacting particles undergoes a major change involving all of its constituents.[1] One of the most striking features of spontaneous fission (or indeed of fission induced by low- and medium-energy projectiles) is the pronounced asymmetry in the mass yield distribution. The fissioning nucleus separates almost always into two unequal parts, with one approximately 1.5 times heavier than the other. Attempts to explain this asymmetry have included a variety of elaborate approaches.[2,3] In this paper we propose a new explanation of the mass asymmetry, based on elementary principles and on very simple, natural assumptions.

Basic assumptions.—The nuclear system and the processes which it undergoes are generally determined by a many-body Hamiltonian, containing (at least) one-body and two-body operators:

$$H=\sum_{i=1}^{A} T_i+\sum_{i<j}^{A} V_{ij}. \tag{1}$$

The process of fission, in particular, is described by a time-dependent solution of this problem. The actual solution of the Schrödinger equation (assuming the Hamiltonian known) is impossible, and approximations need to be made. The liquid-drop model (LDM) describes the nucleus as a homogeneous, nonviscous, incompressible, charged liquid, which is subjected to an irrotational hydrodynamical flow. The surface which determines the instantaneous shape of the nucleus plays the role of dynamical variable,[3,4] and the nuclear potential energy is proportional to the area of this surface. In the present treatment, this surface (or the overall spatial nuclear shape) is still of primary significance, but in addition to it we consider the existence of an intrinsic structure, which reflects the internal nucleonic motion, and represents the individual particles degrees of freedom. The exact nature of this intrinsic structure, and its relation to the Hamiltonian (1) and the nuclear shape, is a question of great complexity. We must make simplifying assumptions, based on physical considerations, to provide for a workable model. We assume that the intrinsic state describes a system of independent particles moving in a common potential. This is motivated by the observation that, reflecting the exclusion principle, nucleons have a long mean free path in nuclear matter. This intrinsic determinantal state is a time-dependent solution of a self-consistency equation, giving a stationary expectation value to the Hamiltonian (1), reproducing the nuclear density of the instantaneous shape, and varying continuously with time. Of these assumptions, the only one that we shall actually exploit here[5] is the continuity of the intrinsic structure.

The common average potential $V(s)$ in which the nucleons move, and which represents their mutual interaction, is regarded as a simple function of the nuclear shape s. It is uniquely defined by the shape s, by assuming, for example, standard depth and surface diffuseness. In addition to the single-particle potential, we must specify the orbits which are actually occupied. This is determined by the assumption of continuity: If for a given shape s a set $\{\chi_i(s); i=1,\cdots,A\}$ of states are occupied, then under a variation[6] δs, the states $\{\chi_i(s+\delta s); i=1,\cdots,A\}$ will be occupied, where

$$\chi_i(s) = \lim_{\delta s=0} \chi_i(s+\delta s). \tag{2}$$

If $\chi_i(s)$ belongs to a nondegenerate eigenvalue of $V(s)$, the continuity of the eigenvalues is sufficient to determine the corresponding occupied state. The operative difference between the LDM and the present approach now becomes apparent. It can be shown[7] that the LDM nuclear potential energy is almost exactly reproduced by assuming that the lowest single-particle states are invariably occupied. In the present approach the level occupation is determined by the initial configuration of the system, regardless of the single-particle energies. Thus, if the system has initially the shape s_0, with A nucleons occupying the (presumably lowest) levels $\chi_{i_1}(s_0), \cdots, \chi_{i_A}(s_0)$, than for any shape s, it should be described as having nucleons occupying the (not necessarily lowest) levels $\chi_{i_1}(s), \cdots, \chi_{i_A}(s)$, where $\chi_i(s)$ is continuously connected to $\chi_i(s_0)$. This single-particle rearrangement primarily modifies the LDM nuclear potential energy of each shape s, by adding to it a positive increment $\Delta E(s)$. The traditional LDM saddle-point shape is symmetric under L_z and R_z (rotation around the fission direction, and reflection through a plane perpendicular to it). For these shapes the rearrangement basically involves the transfer of particles from antisymmetric states of low m to symmetric states of high m, and generally fewer modes. Detailed quantitative studies[5] show that because of the different nature of the states, $\Delta E(s)$ is a strongly decreasing function of asymmetry in the shape. In fact, the variation of $\Delta E(s)$ as a function of shape asymmetry is considerably larger than that of the difference between the Coulomb energy and the unmodified LDM nuclear potential energy. Thus, the modified energy surface has an asymmetric saddle-point shape. The dynamic solution of the modified problem is as complex as that of the unmodified one. However, we may investigate the essential features of such a solution through noting that the system will ideally prefer shapes (and paths) for which $\Delta E(s)$ is minimal. To do this we consider the fission limit, in which the potential approaches asymptotically two separate, uncoupled regions, and the single-particle states are described as pairs of strictly independent functions $(\varphi^{(1)}, \varphi^{(2)})$. Clearly, for $(\varphi^{(1)}, \varphi^{(2)})$ to be an eigenstate of the combined potential, $\varphi^{(1)}$ and $\varphi^{(2)}$, individually, must be eigenstates of the corresponding separate potentials, belonging to the same eigenvalues. Since such a degeneracy is an exceptional occurrence, we see that particles become generally localized completely in one of the fragments, although which one they will localize in may depend on the way the limit is approached. Consider, in particular, paths which maintain throughout the R_z symmetry. (This is done purely for reasons of convenience in mathematical analysis.) An initial independent-particle configuration with A_+ lowest states with $\gamma_z = +1$, and A_- ($A_- \leq A_+$, say) lowest states with $\gamma_z = -1$ (consistent with other quantum numbers), will become

$$\psi = \prod_{i=1}^{A_-} (\varphi_i, -\varphi_i) \prod_{i=1}^{A_+} (\varphi_i, \varphi_i).$$

The pair of states $(\varphi_N, -\varphi_N)$, (φ_N, φ_N) are equivalent, as can be seen by a simple transformation, to $(\varphi_N, 0)$, $(0, \varphi_N)$; namely, two particles localized in the two different fragments. This applies to the lowest A_- pairs. Upon expanding the remaining $A_+ - A_-$ states, we see that they do not correspond to any definite localization (or, equivalently, mass division), but to a linear combination of terms, in each of which N particles are localized in one of the fragments and $A_+ - A_- - N$ in the other, with $A_- \leq N \leq A_+$. The single-particle energies in each of the fragments go down (up) when its spatial dimensions increase (decrease). It is easy to see that $\Delta E(s)$ is smallest, when all $A_+ - A_-$ are localized in one fragment, and the over-all sizes of the fragments are adjusted to reproduce in both the same standard density. With the help of this argument we can summarize the dynamic contents of the modified system, by the following ideal, strikingly simple rule: Under the assumption of continuous evolution of independent nucleonic wave functions, a system having in its ground state A_+ nucleons in eigenstates with $\gamma_z = +1$ and A_- with $\gamma_z = -1$ will divide into two systems, moving apart along the z axis, containing A_+ and A_- nucleons, respectively. In particular, we may draw immediate conclusions concerning the mass yield distribution in spontaneous fission under various hypothetical assumptions.

(A) Initial spherical potential.—The spatial nucleonic single-particle wave functions in a spherical potential are characterized by the quantum numbers n, l, m:

$$\psi_{nlm} = F_{nl}(r) P_m^l(\cos\theta) e^{im\varphi}. \quad (4)$$

Under R_z, $\varphi \to -\varphi$ and $\theta \to -\theta$. Hence

$$R_z \psi_{nlm} = \gamma_z \psi_{nlm} = (-1)^{l-m} \psi_{nlm}. \quad (5)$$

Since l is an integer, there is an odd number, $2l+1$, of states in an l multiplet. For any l, $l+1$ of these states have $\gamma_z = +1$ and only l have $\gamma_z = -1$. Therefore there are always more states with $\gamma_z = +1$ than there are with $\gamma_z = -1$, the difference being equal to the number of occupied l multiplets in the initial system.

(B) Initial deformed potential, symmetric under R_z.—In a deformed initial potential, l is not a good quantum number, and the determination of $A_+ - A_-$ (which determines the mass-number difference between the fragments) is not as straightforward as in the spherical case. It is easy to see that, because of the difference in boundary conditions for symmetric and antisymmetric states on the x-y symmetry plane, $A_+ - A_-$ is a monotonically increasing function of the nuclear cross section on this plane. Hence, the asymmetry in the mass yield distribution in spontaneous fission is a direct measure of both the sign and the magnitude of the intrinsic nuclear ground-state deformation. Qualitatively, for any given nucleus with different hypothetical deformations,

$$(A_H - A_L)_{\text{oblate}} > (A_H - A_L)_{\text{spherical}} > (A_H - A_L)_{\text{prolate}}, \quad (6)$$

where A_H and A_L are the mass numbers of the heavy and light fragments, respectively.

(C) Effect of nucleonic spin.—It is well established that in nuclei there is a strong spin-orbit coupling. The effect of this coupling is to make the half-integer $\vec{j} = \vec{l} + \vec{s}$ a good quantum number, along with the half-integer m_j (rather than m and m_s). An eigenstate $|nljm_j\rangle$ is

$$|nljm_j\rangle = \begin{bmatrix} l & \frac{1}{2} & j \\ m_j - \frac{1}{2} & \frac{1}{2} & m_j \end{bmatrix} \psi_{nl\,m_j - \frac{1}{2}} \chi_{\frac{1}{2}} + \begin{bmatrix} l & \frac{1}{2} & j \\ m_j + \frac{1}{2} & -\frac{1}{2} & m_j \end{bmatrix} \psi_{nl\,m_j + \frac{1}{2}} \chi_{-\frac{1}{2}}, \quad (7)$$

where $\psi_{nl\,m_j \pm \frac{1}{2}}$ are the spatial functions of Eq. (4); $\chi_{\pm\frac{1}{2}}$, spinors; and the bracketed symbols, Clebsch-Gordan coefficients. The states $|nljm_j\rangle$, therefore, are not invariant under R_z. Rather, they are a combination of states with $\gamma_z = +1$

and $\gamma_z = -1$, with probabilities

$$C_{(nljm_j)}^{(\gamma_z)} = \begin{bmatrix} l & \frac{1}{2} & j \\ m_j \mp \frac{1}{2} & \pm\frac{1}{2} & m_j \end{bmatrix}^2$$

$$\text{for } \gamma_z = (-1)^{l-m_j \pm \frac{1}{2}}. \quad (8)$$

The initial nuclear ground state, inasmuch as it is described as a system of independent particles, is therefore a linear combination of states with different values of A_+ and A_-. The total probability for a certain A_+ (and the complement A_-) is simply

$$P(A_+,A_-) = \sum_{\substack{(\alpha_1\cdots\alpha_{A_+}) \\ (\beta_1\cdots\beta_{A_-})}} \left\{\prod_{\lambda=1}^{A_+} C_{\alpha_\lambda}^{(+)}\right\} \times \left\{\prod_{\mu=1}^{A_-} C_{\beta_\mu}^{(-)}\right\}, \quad (9)$$

where the summation extends over all partitions of the A-nucleon system into an A_+- and an A_--nucleon subsystems. Clearly, therefore, the effect of the spin-orbit coupling is to introduce an intrinsic width into the mass distribution. Applying the central limit theorem to this distribution, we rewrite it as

$$P(A_+,A_-) = \left(\frac{1}{2\pi\Gamma}\right)^{1/2} \exp\left[-\frac{1}{2\Gamma}(A_+ - \langle A_+\rangle)^2\right], \quad (10)$$

where

$$\langle A_\pm\rangle = \sum_{\alpha=1}^{A} C_\alpha^{(\pm)} \quad (11)$$

and

$$\Gamma = \sum_{\alpha=1}^{A} C_\alpha^{(+)} C_\alpha^{(-)}. \quad (12)$$

For spontaneous fission, the relative abundance of a fragment with mass number A_F is proportional to $P(A_F, A-A_F) + P(A-A_F, A_F)$. Normalizing the distribution of A_F to 200%, we have, finally,

$$\rho(A_F) = \left(\frac{1}{2\pi F}\right)^{1/2} \left\{\exp\left[-\frac{1}{2\Gamma}(A_F - \langle A_+\rangle)^2\right] + \exp\left[-\frac{1}{2\Gamma}(A_F - \langle A_-\rangle)^2\right]\right\}. \quad (13)$$

As a preliminary example we give in Fig. 1 the theoretical and experimental situation in the spontaneous fission of Cf^{252}, where the fit is essentially made with no free parameters.

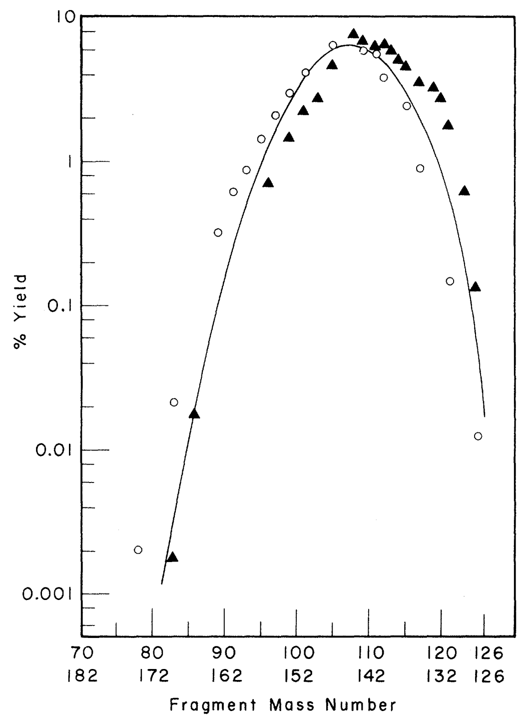

FIG. 1. The theoretical mass yield distribution for the spontaneous fission of Cf^{252}, and the experimental data (not corrected for prompt neutron emission). The circles and triangles represent light and heavy fragments, respectively. The nucleus is assumed to be deformed, with a major-to-minor axis ratio of 1.4 (representing a prolate deformation and a positive quadrupole moment). This results in a shift (towards symmetry) of the peak of the mass distribution by approximately 5 mass units relative to the spherical case.

Obviously, the treatment presented above calls for various modifications, in particular when fission through excited states is to be considered. Nevertheless, it is important to note that the qualitative approach is, in fact, sufficient for understanding the basic features of this phenomenon.

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NUCLEAR STRUCTURE OF Ca^{40} AND ELASTIC SCATTERING OF 750-MeV ELECTRONS

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It is shown that inclusion of the short-range nucleon-nucleon correlations in the usual shell-model description can provide adequate agreement with the known results of the electron elastic-scattering cross section and can make some interesting predictions.

Recently Bellicard et al.[1] published some excellent experimental data on the elastic scattering of 750-MeV electrons from calcium isotopes. It was found that a charge distribution $\rho_0(r)$ obtained by analyzing the scattering data at 250 MeV was quite inadequate at 750 MeV to explain the experimental results beyond scattering angle θ greater than 35°, i.e., in the region of large momentum transfer. An oscillating function, $\Delta\rho_0(r)$, had to be added to the charge distribution $\rho_0(r)$ to obtain a good fit at 750 MeV. We suggest in this note that such a modulating factor arises from the presence of short-range nucleon-nucleon correlations in the Ca^{40} ground-state wave function.

The usual shell-model wave function of a closed-shell nucleus is a Slater determinant of single-particle functions determined in a central potential well. In the case of Ca^{40}, the shell-model wave function is taken as the closed $1s$, $1p$, $2s$, and $1d$ shells. Such a wave function has few high-momentum components but enough low-momentum components to explain experimental results that involve momenta $p \lesssim p_F$, the Fermi momentum. For example, this wave function can provide an adequate explanation for the elastic electron scattering at 250 MeV since at this energy only low-momentum components are being studied. However, at 750-MeV electron energy and $\theta > 35°$, high-momentum components are important and the usual shell model is expected to break down. High-momentum components can arise from the strong short-range repulsion and the attractive part just outside the repulsive core in the nucleon-nucleon potential.[2] A radical approach to modify the wave function would be to do a Brueckner-type calculation for this finite nuclear system. Since such an approach is extremely difficult and has many uncertainties, we choose the phenomenological method suggested by Jastrow.[3]

The modified wave function, ψ, for the ground state of Ca^{40} is chosen to have the form

$$\psi(r_1 \cdots r_{40}) = N \prod_k \varphi_k(r_k) \prod_{i>j} f(r_{ij}), \tag{1}$$

where $\varphi_k(r_k)$ are the harmonic-oscillator-type single-particle wave functions and $f(r_{ij})$ is the Jastrow-type factor with the properties that

$$\lim_{r_{ij}\to\infty} f(r_{ij}) = 1 \text{ and } \lim_{r_{ij}\to 0} f(r_{ij}) = 0.$$

Explicitly $f(r_{ij})$ is chosen to be

$$f(r_{ij}) = 1 - \exp(-\beta^2 r_{ij}^{\ 2}), \tag{2}$$

where β is a parameter to be determined. The elastic-scattering cross section for electrons of energy E on a nucleus with charge Z in the Born approximation is well known to have the form[4]

$$\frac{d\sigma}{d\Omega} = \frac{Ze^2 \cos^2(\theta/2)}{4E^2 \sin^4(\theta/2)} \frac{|F(q^2)|^2}{1 + (2E/M)\sin^2(\theta/2)}, \tag{3}$$

where θ is the scattering angle, M is the mass of the target nucleus, and $\hbar = c = 1$. $F(q^2)$ is the charge form factor of the nucleus and is given as

$$F(q^2) = \frac{1}{Z}\int \psi^* \sum_j^{\text{protons}} \exp(i\vec{q}\cdot\vec{r}_j)\psi d\tau. \tag{4}$$

Because of the complexity of ψ, this expression for $F(q^2)$ can be evaluated exactly only for simple systems like He^4. But for Ca^{40}, we use the cluster expansion as given by Iwamoto and Yamada.[5] Here only one- and two-particle cluster