# EXTENSION OF THE LOW SOFT-PHOTON THEOREM*

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The Low theorem is applied to the radiative cross section for unpolarized particles. It is shown that the first two terms of an expansion in the photon energy depend on the unpolarized, nonradiative cross section only.

Low[1] has shown that the first two terms of a radiative cross section, expanded in powers of the photon energy $k$, can be exactly determined given the amplitude for the nonradiative process. Specifically, when the cross section is expanded in powers of the energy loss $k$,

$$\sigma = \sigma_0/k + \sigma_1 + k\sigma_2 + \cdots, \quad (1)$$

a unique value for

$$\sigma_0 = \lim_{k\to 0} k\sigma$$

and

$$\sigma_1 = \lim_{k\to 0} \frac{d(k\sigma)}{dk}$$

can be obtained from knowledge of the nonradiative amplitude.

Our extension applies to the cross section when the charged particles have spin: The terms $\sigma_0$ and $\sigma_1$ for the cross section summed and averaged over the spins of those particles bearing charge or magnetic moment are uniquely determined by the nonradiative cross section summed and averaged over the spins of the same particles. This statement is obvious for $\sigma_0$, which is directly proportional to the nonradiative cross section in any case. The term $\sigma_1$, however, contains interference terms between the elastic amplitude and its derivative with respect to physical variables; so it is appropriate to inquire whether or not phase information can be obtained from measurement of $\sigma_1$. We will show that the interference between electric and magnetic terms, which $\sigma_1$ also contains, combines with the above to produce the result that $\sigma_1$ depends only on the derivatives of the unpolarized cross section.

Low's result depends on the fact that the radiative amplitude contains contributions from two types of terms: terms in which the photon is radiated from an external line, or, equivalently, terms with a pole in the variable $(p-k)^2$, where $p$ is the momentum of any particle, and terms in which the photon is radiated from an internal line, having no poles of this type (see Fig. 1). These pole terms are responsible for infrared divergence [$O(1/k)$ behavior], while the nonpole terms are finite at zero photon energy.

The residue of the pole terms, which contribute to $\sigma_0$, factors into the product of the physical nonradiative amplitude and the sum of the amplitudes for electric radiation of all the charged particles, a result that can be obtained classically. The next term in an expansion of the amplitude would involve unphysical off-mass-shell contributions from both the scattering and the photon emission, but the result of Low is that gauge invariance requires these to vanish. We now present our formulation of the Low thorem and illustrate this point.

We denote the radiative amplitude by $T_\gamma(\epsilon, k; \cdots)$ which we divide into internal and external parts, $T_\gamma = T_\gamma^{\text{ext}} + T_\gamma^{\text{int}}$, as discussed above. The nonradiative amplitude will be denoted by $T(\cdots)$, where $\cdots$ represents the momenta and polarizations of the particles.

The $O(k^0)$ terms are of three types: (1) $k$-independent terms; (2) pole terms of the general form $(\epsilon\cdot p_a/k\cdot p_a)k_\mu M_a{}^\mu$, where $M_a{}^\mu$ is $k$ independent; and (3) explicitly gauge-invariant pole terms which may be written in the form $\epsilon_\mu k_\nu \sum_a N_a{}^{\mu\nu}/k\cdot p_a$, where $N_a{}^{\mu\nu}$ is $k$ independent and $k_\mu k_\nu \sum_a N_a{}^{\mu\nu}/k\cdot p_a = 0$. The pole terms come only from $T_\gamma{}^{\text{ext}}$, while the $k$-independent terms may come from either $T_\gamma{}^{\text{ext}}$ or $T_\gamma{}^{\text{int}}$. Adler and Dothan[2] have pointed out that the $k$-independent terms are completely deter-

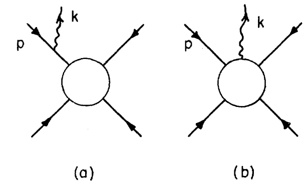


FIG. 1. Radiative-amplitude diagrams. (a) Pole erm. (b) Nonpole term.

mined by the gauge-invariance requirement $T_\gamma(k,k;\cdots)=0$, the sum of the $k$-independent terms being given by

$$-\sum_a \epsilon_\mu M_a^{\ \mu}. \tag{2}$$

Now consider $T_\gamma^{\text{ext}}$ for scalar particles only:

$$T_\gamma^{\text{ext}}(\epsilon,k;\cdots)=\sum_a Q_a \frac{\epsilon\cdot p_a}{k\cdot p_a} T(\cdots,p_a-k,\cdots). \tag{3}$$

Here $Q_a$ is the charge of the $a$th particle, and all particles except the photon are considered incoming, so that $\sum_a p_a = k$. We now define a procedure for extrapolation from a $T$ with a physically realizable set of momenta $\{p_a'\}$ such that $\sum_a p_a' = 0$ and $p_a'^{\,2} = M_a^2$, to $T$'s of the form given in Eq. (3), where one momentum is unphysical, and the rest the same as in $T_\gamma$. Let $p'(k)=p_a-\xi_a(k)$. Then the vectors $\xi$ must have the properties $\sum\xi_a(k)=k$, $p_a\cdot\xi_a=0$, $\xi_a(0)=0$, $(\partial\xi_a/\partial k_\mu)_{k_\mu=0}$ finite or zero.

Then we can define the expansion

$$T(\cdots,p_a-k,\cdots)$$
$$=T(p')+\sum_b \xi_b\cdot\frac{\partial}{\partial p_b}T(p)\Big|_{p=p'}$$
$$-k\cdot\frac{\partial}{\partial p_a}T(p)\Big|_{p=p'}+O(k^2). \tag{4}$$

The prescription then is that $T$ is expressed in terms of the momenta $p_a$ which are differentiated as independent variables, then evaluated with the momenta $p_a'$. It is possible for the $\xi$'s to be defined so that scalar variables are the same to first order in $k$ whether expressed in terms of $p$ or $p'$; so for scalar amplitudes at least, the second term in Eq. (4) can be neglected. Thus the choice of variables to an extent determines the extrapolation, as well as the form that the Low theorem takes.[3] Substituting Eq. (4) into Eq. (3) and using the expression (2) to determine the $k$-independent terms, we have

$$T_\gamma(\epsilon,k;\cdots)=\sum_a Q_a\frac{\epsilon\cdot p_a}{k\cdot p_a}\left[T(p')+\sum_b\xi_b\cdot\frac{\partial}{\partial p_b}T(p')\right]-\sum_a Q_a\left[\frac{\epsilon\cdot p_a}{k\cdot p_a}k\cdot\frac{\partial}{\partial p_a}-\epsilon\cdot\frac{\partial}{\partial p_a}\right]T(p').$$

For convenience in the following, we will define the differential operator

$$D_a(k)=\frac{p_a}{k\cdot p_a}k\cdot\frac{\partial}{\partial p_a}-\frac{\partial}{\partial p_a}.$$

Then the property $k\cdot D_a(k)=0$ and charge conservation expressed as $\sum Q_a=0$ imply the required gauge-invariance property $T_\gamma(k,k,\cdots)=0$. The properties $\sum_b\xi_b\cdot(\partial/\partial p_b)p_a^{\,2}=0$ and $D_a(k)p_a^{\,2}=0$ show that derivatives of $T(\cdots)$ with respect to masses do not contribute.

When spin is involved, we must include magnetic terms $O(k^0)$ from the photon vertex. We should also verify that $T$, which is now a sum of invariants involving polarization tensors or spinors and momenta multiplied by scalar functions, involves only the invariants present in the physical nonradiative process. This is done[2] by splitting $T$ into physical and nonphysical parts [by $(\not p\pm m)/2m$ projection operators in the spin-$\frac{1}{2}$ case] and finding that the only $O(k^0)$ contribution from the unphysical part of $T_\gamma^{\text{ext}}$ is independent of $k$, and thus included in the expression (2).

We demonstrate our result by considering the interference of the $O(k^0)$ term for a Dirac particle with the total $O(1/k)$ contribution and summing over the spins of the Dirac particle. If it has momentum $p$ and charge $Q$ and we define $\mathcal{T}(p)u(p)=T(p)$, then the Low theorem, neglecting $O(k^0)$ contributions from the other charged particles for the moment, takes the form

$$T_\gamma(\epsilon,k;p,\cdots)=\sum_a Q_a\frac{\epsilon\cdot p_a}{k\cdot p_a}\left[\mathcal{T}(p')+\xi\cdot\frac{\partial}{\partial p}\mathcal{T}(p')\right]u(p)$$
$$-\frac{1}{2k\cdot p}\mathcal{T}(p)\{Q\not k\not\epsilon-\lambda(\not p+m)[\not\epsilon,\not k]\}u(p)-Q\epsilon\cdot D(k)\mathcal{T}(p')u(p)+O(k), \tag{5}$$

where $4m\lambda$ is the anomalous magnetic moment. Let $A=\sum_a Q_a\epsilon\cdot p_a/k\cdot p_a$ and $\overline{\mathcal{T}}=\gamma_0\mathcal{T}^\dagger$. Then, provided that $\epsilon$ is real,

$$\sum_{\text{spin}} |T_\gamma|^2 = A^2 \Big\{ \mathcal{T}(p')(\not{p}+m)\overline{\mathcal{T}}(p') + \mathcal{T}(p')(\not{p}+m)\xi\cdot\frac{\partial}{\partial p}\overline{\mathcal{T}}(p') + \Big[\xi\cdot\frac{\partial}{\partial p}\mathcal{T}(p')\Big](\not{p}+m)\overline{\mathcal{T}}(p')\Big\}$$
$$-QA\Big\{\mathcal{T}(p')(\not{p}+m)\epsilon\cdot D(k)\overline{\mathcal{T}}(p') + [\epsilon\cdot D(k)\mathcal{T}(p')](\not{p}+m)\overline{\mathcal{T}}(p')$$
$$+\mathcal{T}(p')\frac{(\not{p}+m)\not{\epsilon}\not{k}+\not{k}\not{\epsilon}(\not{p}+m)}{2k\cdot p}\overline{\mathcal{T}}(p')\Big\} + O(k^0). \qquad (6)$$

The terms from the anomalous magnetic moment have cancelled. Now we observe that

$$\frac{(\not{p}+m)\not{\epsilon}\not{k}+\not{k}\not{\epsilon}(\not{p}+m)}{2k\cdot p} = \frac{\epsilon\cdot p}{k\cdot p} - \not{\epsilon} = \epsilon\cdot D(k)(\not{p}+m). \qquad (7)$$

Inserting Eq. (7) in Eq. (6) and replacing $(\not{p}+m)$ by $(\not{p}'+m)$ using the relation

$$(\not{p}'+m) = (\not{p}+m) - \xi\cdot\frac{\partial}{\partial p}(\not{p}+m)|_{p=p'}, \qquad (8)$$

we obtain

$$\sum_{\text{spin}} |T_\gamma|^2 = A^2\Big[\mathcal{T}(p')(\not{p}'+m)\overline{\mathcal{T}}(p') + \xi\cdot\frac{\partial}{\partial p}\mathcal{T}(p')(\not{p}'+m)\overline{\mathcal{T}}(p')\Big] - QA\epsilon\cdot D(k)\mathcal{T}(p')(\not{p}'+m)\mathcal{T}(p') + O(k^0). \qquad (9)$$

Thus the nonradiative amplitude only enters in its unpolarized form. Equation (7), which is fundamental to this result, is essentially a relation involving the vertex function and propagator to lowest order in photon energy. The analogous relation for vector particles also holds, and we conjecture that it is true for any spin.

Since the result, Eq. (9), was derived independently of the other spins, we may sum them also, obtaining

$$\sum_{\text{spins}} |T_\gamma|^2 = A^2\Big(1+\sum_b \xi_b\cdot\frac{\partial}{\partial p_b}\Big)\sum_{\text{spins}}|T|^2 - A\sum_b Q_b\epsilon\cdot D_b(k)\sum_{\text{spins}}|T|^2. \qquad (10)$$

Now, since

$$\sum_{\text{spins}}|T|^2$$

depends only on scalar invariants, we may, as we discussed before, define the $\xi$'s so that the $\sum_b \times \xi_b\cdot(\partial/\partial p_b)$ term vanishes. In that case we obtain the final form

$$\sum_{\text{spins}} |T_\gamma(\epsilon,k;p)|^2 = \sum_a Q_a\frac{\epsilon\cdot p_a}{k\cdot p_a}\sum_b Q_b\Big[\frac{\epsilon\cdot p_b}{k\cdot p_b} - \epsilon\cdot D_b(k)\Big]\sum_{\text{spins}}|T(p')|^2. \qquad (11)$$

In conclusion, we have shown that, within the photon energy range for which the Low theorem is useful, it is not possible to obtain any more information about a nonradiative process by studying the behavior of the unpolarized radiative process. Conversely, the Low contributions to an unpolarized radiative process can be deduced from the nonradiative unpolarized cross section alone.

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[1]F. E. Low, Phys. Rev. 110, 974 (1958).

[2]S. L. Adler and Y. Dothan, Phys. Rev. 151, 1267 (1966).

[3]The point is that the variables $p'$ and $p$ are not and cannot be the same to $O(k)$. Thus the point at which $\sigma_0$ is evaluated in (1) is not unambiguously specified. A change in the choice of the $p'$ induces an $O(k^0)$ change in $\sigma_0/k$, this change being compensated by a corresponding change in $\sigma_1$. When, as is the case with unpolarized cross sections, the function can be fully defined in terms of scalar invariants, the same values of these invariants can be used for radiative and nonradiative process. Note, however, that even in this case changes in the choice of scalar invariants change $\sigma_1$, with compensating changes in the $O(k^0)$ part of $\sigma_0/k$.