CALCULATION OF π-π PHASE SHIFTS*

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Beginning with Chew and Mandelstam,[1,2] many authors have studied once-subtracted dispersion relations (D.R.'s) for the π-π partial-wave amplitudes. Unfortunately, the requisite integrals converge so slowly that the treatment of distant singularities is highly problematic. In this paper, we alleviate the problems associated with the distant singularities by subtracting each D.R. either twice or l times, whichever is greater. We present a method for approximately solving the resulting D.R.'s when the S-wave scattering lengths a_I are given. We then use the a_I predicted by Weinberg[3] and by Schwinger[4] to calculate S-wave phase shifts for $E_{\text{c.m.}}$ up to 900 MeV. Unlike Fulco and Wong[5] and Brown and Goble,[6] we obtain solutions which agree with the values reported by Walker et al.[7]

Our conventions include $m_\pi = \hbar = c = 1$, $\nu = \frac{1}{4}S - 1$; we normalize the amplitudes $A^I(\nu, \cos\theta)$ such that $A^{(l)I}(\nu) = (1+1/\nu)^{\frac{1}{2}} \exp(i\delta_l^I) \sin\delta_l^I$. Bose symmetry implies that even (odd) l contribute only for even (odd) I.

We assume that every $A^{(l)I}$ is a real analytic function of ν except for cuts from $-\infty$ to -1 and from 0 to $+\infty$. Two subtractions are presumed to ensure rapid convergence for the dispersion integrals. Noting that $A^{(0)I}(0) = a_I$, we write

$$\mathrm{Re}A^{(0)I}(\nu) = a_I + \frac{A^{(0)I}(\nu_0) - a_I}{\nu_0}\nu + \frac{\nu(\nu-\nu_0)}{\pi} \mathrm{P}\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}A^{(0)I}(\nu')}{\nu'(\nu'-\nu_0)(\nu'-\nu)}, \tag{1a}$$

$$\mathrm{Re}A^{(1)I}(\nu) = \frac{A^{(1)I}(\nu_0)}{\nu_0}\nu + \frac{\nu(\nu-\nu_0)}{\pi} P\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}A^{(1)I}(\nu')}{\nu'(\nu'-\nu_0)(\nu'-\nu)}, \tag{1b}$$

$$\mathrm{Re}A^{(2)I}(\nu) = \frac{\nu^2}{\pi} \mathrm{P}\int_{-\infty}^{\infty} d\nu' \frac{\mathrm{Im}A^{(2)I}(\nu')}{\nu'^2(\nu'-\nu)}, \text{ etc.} \tag{1c}$$

Since $A^{(l)I}(\nu)$ vanishes like ν^l as $\nu \to 0$, we can subtract the D.R.'s for all higher partial waves $l > 2$ times at $\nu = 0$ without introducing any more constants.

As usual, we approximate the amplitudes $A^I(\nu, \cos\theta)$ with some number N of partial waves. We then use crossing symmetry together with the Legendre series to express each $\mathrm{Im}A^{(l)I}(\nu)$ for $\nu \leqslant -1$ in terms of all the $\mathrm{Im}A^{(l')I'}(\nu')$ with $\nu' \geqslant 0$.[1,2] Under these approximations, the D.R.'s give each $\mathrm{Re}A^{(l)I}(\nu)$ as an explicit function of subtraction constants and all the $\mathrm{Im}A^{(l')I'}(\nu')$ with $\nu' \geqslant 0$. Unitarity implies another explicit relation between $\mathrm{Re}A^{(l)I}$ and $\mathrm{Im}A^{(l)I}$, namely,

$$\mathrm{Re}A^{(l)I}(\nu) = \{\mathrm{Im}A^{(l)I}(\nu)[(1+1/\nu)^{\frac{1}{2}} - \mathrm{Im}A^{(l)I}(\nu)]\}^{\frac{1}{2}}. \tag{2}$$

Equation (2) is exact from the elastic threshold at $\nu = 0$ up to the first inelastic threshold at $\nu = 3$ ($2\pi \to 4\pi$), and experience with pion-nucleon scattering suggests that it is approximately valid up to $\nu \sim 10$.

We specify the subtraction point ν_0 to be $\nu|_{\text{sym pt}} = -\frac{2}{3}$. Then crossing symmetry implies the relations[2]

$$\tfrac{1}{5}A^0(\nu_0, 0) = \tfrac{1}{2}A^2(\nu_0, 0) \equiv -\lambda, \tag{3a}$$

$$\frac{1}{2}\frac{\partial A^0}{\partial\nu}\bigg|_{\nu_0,0} = -\frac{\partial A^2}{\partial\nu}\bigg|_{\nu_0,0} = \frac{\partial}{\partial\cos\theta}\left(\frac{A^1}{\nu}\right)\bigg|_{\nu_0,0} \equiv \lambda_1. \tag{3b}$$

There is a total of five subtraction constants in the D.R.'s (1), but the crossing relations (3) enable us to express all five constants in terms of the a_I and integrals which are presumed to converge rapidly. Thus, when the a_I are known, Eqs. (1)-(3) constitute $3+N$ simultaneous equations for the three unknown subtraction constants and the N functions $\mathrm{Im}A^{(l)I}(\nu)$, $0\leq\nu\lesssim 10$. To solve these equations, we shall represent each $\mathrm{Im}A^{(l)I}(\nu)$ by a multiparameter trial function $F^{(l)I}(\nu;c_j^{(l)I})$ which is a linear function of some number $n^{(l)I}$ of parameters $c_j^{(l)I}$. (For example, we might use

$$F^{(0)I}=\nu^{\frac{1}{2}}\sum_{j=1}^{n^{(0)I}} c_j^{(0)I}\nu^{(j-1)}$$

for $0\leq\nu\leq\Lambda$, a cutoff.) Then every dispersion integral becomes a linear function of the parameters $c_j^{(l)I}$, wherein the coefficients of the $c_j^{(l)I}$ are simply integrals of known functions. For given values of the a_I, it is then straightforward to compute values for the subtraction constants and the $c_j^{(l)I}$ such that the crossing relations (3) are satisfied, and such that the functions $F^{(l)I}(\nu;c_j^{(l)I})$ approximately satisfy the D.R.'s (1) together with unitarity (2) at some finite set of points $\{\nu_n\}$. If the forms of the trial functions and the set of points $\{\nu_n\}$ are suitably chosen, then the resulting functions $F^{(l)I}(\nu;c_j^{(l)I}(a_{I'}))$ will be approximate solutions for the $\mathrm{Im}A^{(l)I}(\nu)$ throughout the region spanned by the points $\{\nu_n\}$. These ideas will be made clearer in the following application.

Let us approximate the amplitudes $A^I(\nu,\cos\theta)$ by sums of S, P, and D waves. In this paper, we calculate only the S waves; we regard the P and D waves as given functions of ρ, f_0, and f_0' parameters for $\nu\geq 0$, and we assume $\mathrm{Im}A^{(2)2}(\nu)$ to be negligible for $\nu\geq 0$. The resonance parameters reported by Rosenfeld et al.[8] are $\nu(\rho)=6.67$, $\Gamma(\rho)=1.02$; $\nu(f_0)=19.7$, $\Gamma(f_0)=0.85$; $\nu(f_0')=29.2$, $\Gamma(f_0')=0.62$. These parameters are to be substituted into Breit-Wigner–type formulas with appropriate threshold behavior; we use[9]

$$\mathrm{Im}A^{(l)I}(\nu)=\frac{(1+1/\nu)^{\frac{1}{2}}\gamma_R^{\,2}\nu^{2l+1}}{16(\nu+1)(\nu-\nu_R)^2+\gamma_R^{\,2}\nu^{2l+1}},\qquad(4)$$

where $\gamma_R^{\,2}\equiv 4(\nu_R+1)^2\Gamma_R^{\,2}/\nu_R^{\,2l+1}$. The f_0 and f_0' resonances are well separated, so we simply use the sum of their contributions for $\mathrm{Im}A^{(2)0}(\nu)$, $\nu\geq 0$.

Next we introduce trial functions $F^{(0)I}(\nu;c_j^{(0)I})$ to represent each $\mathrm{Im}A^{(0)I}(\nu)$, $\nu\geq 0$. The choice of trial functions is fairly arbitrary, but some guide lines are worth noting. The $F^{(0)I}$ are to be defined for $0\leq\nu\leq\Lambda$, where Λ is some cutoff value large enough to justify neglecting higher contributions to the integrals. Each $F^{(0)I}$ must tend to $a_I^{\,2}\sqrt{\nu}$ as $\nu\to 0$. The $F^{(0)I}$ and $(d/d\nu)F^{(0)I}$ should be continuous for $0<\nu<\Lambda$. Lastly, the $F^{(0)I}$ should be especially flexible in the regions where they make their major contributions to the integrals, and wherever the solutions vary rapidly.

In our present calculation, we shall impose the D.R.'s (1) and unitarity (2) at a set of points $\{\nu_n\}$ on the interval $0\leq\nu_n\leq 6$ (276 MeV $\leq E_{\mathrm{c.m.}}\leq$ 730 MeV). We place the cutoff at $\Lambda=20$ (1260 MeV). For $0\leq\nu\leq 20$, we define each $F^{(0)I}$ to be a linear function of six parameters $c_1^{(0)I}$, $\cdots, c_6^{(0)I}$ which represent $a_I^{\,2}$, $\mathrm{Im}A^{(0)I}(\nu_i)$ at $\nu_i=\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$, and 6, and $(d/d\nu)\mathrm{Im}A^{(0)I}|_6$, respectively. We further define the $F^{(0)I}$ to depend on ν in the following ways: for $0\leq\nu\leq\frac{1}{2}$, $F^{(0)I}(\nu)=\nu^{\frac{1}{2}}\times$quadratic; for $\frac{1}{2}\leq\nu\leq\frac{5}{2}$, $F^{(0)I}(\nu)=$quadratic; for $\frac{5}{2}\leq\nu\leq 6$, $F^{(0)I}(\nu)=$cubic. We will obtain solutions which suggest a broad isoscalar resonance somewhat above $\nu=10$. Accordingly, we extrapolate $F^{(0)0}$ above $\nu=6$ by $F^{(0)0}(\nu)=1-\alpha(\beta+\nu)^2/\nu^4$ for $6\leq\nu\leq 20$, where α and β are determined by $c_5^{(0)0}$, $c_6^{(0)0}$, and our continuity conditions. We use a linear extrapolation[10] for $F^{(0)2}$: $F^{(0)2}(\nu)=$linear for $6\leq\nu\leq 20$. Finally, we impose continuity on each $F^{(0)I}$ and $d/d\nu F^{(0)I}$ for $0<\nu<\Lambda=20$. These conditions uniquely determine each $F^{(0)I}$ in terms of ν and the corresponding six parameters $c_j^{(0)I}$ defined previously. It is straightforward to construct explicit formulas for the $F^{(0)I}(\nu;c_j^{(0)I})$, and to compute their contributions to all requisite integrals.

The only remaining step in our solution is to impose Eqs. (1) and (2) on the $F^{(0)I}$ at a set of points $\{\nu_n\}$ in such a way that the resulting $F^{(0)I}(\nu;c_j^{(0)I}(a_{I'}))$ will be approximate solutions for the $\mathrm{Im}A^{(0)I}$ over the interval $0\leq\nu\leq 6$. We proceed in the following way. Let the points $\nu_n=\frac{1}{2}n$, $n=1,\cdots,12$.[11] Next we substitute $\pm\{F^{(0)I}(\nu_n)[(1+1/\nu_n)^{\frac{1}{2}}-F^{(0)I}(\nu_n)]\}^{\frac{1}{2}}$ for $\mathrm{Re}A^{(0)I}(\nu_n)$ in the left-hand sides of the D.R.'s (1a), letting the sign of the radical be determined by the sign of the right-hand side, which is now a known function of the $a_{I'}$, the $c_j^{(0)I'}$, and the ρ, f_0, and f_0' parameters. In general, there will be a discrepancy $\Delta^{(0)I}(\nu_n;a_{I'};c_j^{(0)I'})$ between the resulting left- and right-hand sides. For given values of the $a_{I'}$, we compute the values of $c_2^{(0)I'}, \cdots, c_6^{(0)I'}$ which minimize a suitably weighted sum of discrepancies squared; specifically, we minimize the function

$$\Delta^2 \equiv \sum_{I=0,2} \sum_{n=1}^{12} [\Delta^{(0)I}(\nu_n; a_{I'}; c_j^{(0)I'})/\nu_n]^2.$$

Then the resulting functions $F^{(0)I}(\nu; c_j^{(0)I}(a_{I'}))$ are approximate solutions[12] for $\mathrm{Im}A^{(0)I}$ throughout the region $0 \leq \nu \leq 6$. Once the $c_j^{(0)I}(a_{I'})$ are known, we calculate the phase shifts up to $\nu = 10$ (915 MeV) by evaluating $\mathrm{Re}A^{(0)I}(\nu)$ directly from the D.R.'s (1a).

Let us define a parameter L:

$$L \equiv \tfrac{1}{6}(2a_0 - 5a_2). \tag{5}$$

Weinberg[3] and Schwinger[4] have used current algebra and chiral symmetry, respectively, to predict[13]

$$L = g_V^2 m_\pi / 2\pi F_\pi^2 \cong 0.10 m_\pi^{-1}. \tag{6}$$

The ratio a_0/a_2 depends on details of the chiral-symmetry-breaking part of the Lagrangian. Weinberg's hypothesis for the breaking of symmetry leads to $a_0/a_2 = -\frac{7}{2}$, while Schwinger suggests two alternative hypotheses which lead to the ratios $-\frac{3}{2}$ and $-\frac{1}{2}$, respectively.

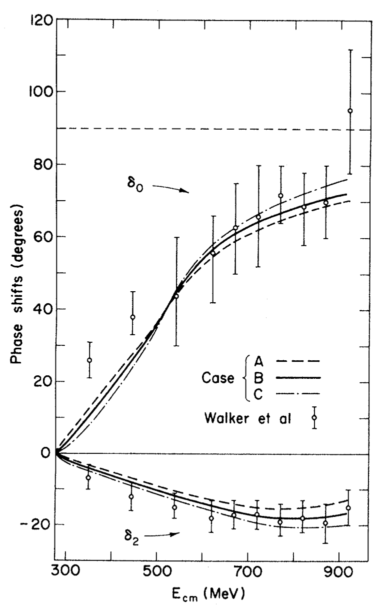

FIG. 1. The phase shifts calculated by method described in text. Cases A, B, and C correspond to $(a_0/a_2, L) = (-\frac{7}{2}, 0.105)$, $(-\frac{3}{2}, 0.100)$, and $(-\frac{1}{2}, 0.095)$, respectively.

For the three pairs of a_I with $(a_0/a_2, L)$ given by A, $(-\frac{7}{2}, 0.105)$; B, $(-\frac{3}{2}, 0.100)$; and C, $(-\frac{1}{2}, 0.095)$, we present in Fig. 1 our solutions for the corresponding S-wave phase shifts δ_I, together with the δ_I reported by Walker et al.[7] The agreement is excellent except for δ_0 below 500 MeV. Walker's δ_I were inferred from $\pi N \to \pi\pi N$ data by assuming a peripheral model for the amplitudes. Our small values for δ_0 near threshold suggest that peripheral π production is not dominant there, so that Walker may have misinterpreted the data below 500 MeV.[14]

As L and a_0/a_2 vary over the ranges $0.085 \leq L \leq 0.115$, $-\frac{7}{2} \leq a_0/a_2 \leq -\frac{1}{2}$, we find that within 3%, the δ_I at 500 and 730 MeV are given by

$$\delta_0(500) = 35.8° + (L - 0.100) \times 230°, \tag{7a}$$

$$\delta_2(500) = -10.2° - (L - 0.100) \times 120° + (a_2 + 0.75L) \times 72°, \tag{7b}$$

$$\delta_0(730) = 64° - (L - 0.100) \times 530°, \tag{7c}$$

$$\delta_2(730) = -17.6° - (L - 0.100) \times 360° + (a_2 + 0.75L) \times 160°. \tag{7d}$$

For $0.090 \leq L \leq 0.105$ and $-\frac{7}{2} \leq a_0/a_2 \leq -\frac{1}{2}$,

$$\delta_0(915) = 72° - (L - 0.100) \times 1200° + (a_2 + 0.75L) \times 160°, \tag{7e}$$

$$\delta_2(915) = -16.8° - (L - 0.100) \times 680° + (a_2 + 0.75L) \times 290°, \tag{7f}$$

within 3% and 2°, respectively.

The dispersion integrals in the D.R.'s (1a) are multiplied by $\nu(\nu - \nu_0) \sim E^4$, so that our solutions become less reliable as E increases. We present in Table I uncertainties in $\delta_I(E)$ which result from uncertainties in the ρ, f_0, and f_0' widths, and also variations in the δ_I

Table I. Changes in $\delta_I(E)$ resulting from changes in the input. Changes listed are averages for cases A, B, and C.

Δ	$\delta_0, \delta_2(500)$	$\delta_0, \delta_2(730)$	$\delta_0, \delta_2(915)$
$\Gamma(p)$: ±20%	∓1.8°, ±0.3°	±2.5°, ±2.1°	±7°, ±5°
$\Gamma(f_0)$: ±20%	±0.3°, ±0.1°	∓0.6°, ±1.0°	∓2°, ±3°
$\Gamma(f_0')$: ±20%	±0.1°, ±0.0°	∓0.2°, ±0.3°	∓1°, ±1°
$\Lambda = 20$: +5	−0.0°, −0.0°	−0.4°, +0.2°	−2°, +1°
−5	−0.2°, −0.1°	+1.0°, −0.5°	+6°, −2°
−10	−0.6°, −0.2°	+5.0°, −0.8°	$\mathrm{Re}A^{(0)I} = \infty$

which result from varying the cutoff Λ. From threshold up to $m_K c^2 = 500$ MeV, the solutions are quite stable, even if we lower the cutoff to $\Lambda = 10$. This energy range is of interest for weak interactions. Equations (7a) and (7b) imply that if $L = 0.100 \pm 15\,\%$ and $-\frac{7}{2} \leq a_0/a_2 \leq -\frac{1}{2}$, then

$$\delta_0(m_K) - \delta_2(m_K) = 46° \pm 7°. \tag{8}$$

Equations (1) and (3) imply that

$$\lambda = -\tfrac{1}{9}(a_0 + 2a_2) + \text{integrals}, \tag{9a}$$

$$\lambda_1 = L + \text{integrals}, \tag{9b}$$

where the integrals are quite small for the solutions obtained in this paper. We present in Table II the values for λ, λ_1, $A^{(2)I}(\nu_0)$, and Δ^2 which correspond to the solutions A, B, and C of Fig. 1.

Our values for the ρ, f_0, and f_0' contributions to

$$\int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A^{(0)I}(\nu')}{\nu'(\nu'-\nu_0)(\nu'-\nu)}$$

are roughly given near threshold for

$$I = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

by

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} \times \frac{(-0.25)}{11+\nu}, \quad \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \times \frac{0.66}{56+\nu}, \quad \text{and} \begin{pmatrix} \frac{2}{3} \\ \frac{2}{3} \end{pmatrix} \times \frac{0.26}{79+\nu}, \tag{10}$$

respectively. However, the Legendre series converges[1] only for $\nu > -9$. Since the poles in Eq. (10) occur for $\nu < -9$, our values for the resonance contributions are open to question, especially the f_0 and f_0' contributions.

Recently Malamud and Schlein[15] have used $\pi N \to \pi\pi N$ data and a modified peripheral model to infer δ_I somewhat different from those of Fig. 1. They report an isoscalar resonance near 750 MeV, with $\sin^2\delta_0$ decreasing to $\frac{1}{2}$ near 820 MeV ($\nu \sim 8$). If we seek a self-consistent solution of this type by placing the cutoff at $\Lambda = 10$, we obtain

$$\delta_0(730) \cong 70° - (L-0.100)\times 350 - (L-0.100)^2 \times 5000°, \tag{11a}$$

$$d\delta_0/dE\,|_{730} \cong 0.2°/\text{MeV}, \tag{11b}$$

for $0.08 \leq L \leq 0.12$, $-\frac{7}{2} \leq a_0/a_2 \leq -\frac{1}{2}$. Thus it may be possible to obtain a class of solutions for δ_0 with a resonance near 750 MeV. This possibility is being studied further.[16] On the other hand, Malamud and Schlein's δ_2 is inconsistent with the a_I considered in this paper, and its energy dependence seems implausible.

Table II. Selected characteristics of solutions A, B, and C.

	λ	λ_1	$A^{(2)0}(\nu_0)$	$A^{(2)2}(\nu_0)$	Δ^2
A	-0.0036	0.104	9.4×10^{-4}	1.8×10^{-4}	2.2×10^{-4}
B	0.0085	0.103	9.4×10^{-4}	1.3×10^{-4}	3.7×10^{-4}
C	0.0198	0.101	9.5×10^{-4}	9.2×10^{-5}	4.6×10^{-4}

Obviously the P wave is of great interest. In future work, we can take the subtraction constant $A^{(1)1}(\nu_0) \cong -(2/9)\lambda_1$ from our present work and predict the P-wave and ρ parameters, or we can assume that $\operatorname{Re}A^{(1)1} = 0$ at 769 MeV and predict the ρ width and λ_1. More ambitious predictions can also be attempted.[16] Such work is in progress.

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[10]For solutions A, B, and C of Fig. 1, the derivatives $(d/d\nu)F^{(0)2}|_6$ are quite small; they equal 0.007, 0.012, and 0.016, respectively. Thus a simple linear extrapolation is sufficiently consistent with the solutions to justify its use.

[11]The number of points in the set $\{\nu_n\}$ should exceed the number of free parameters in $F^{(0)I}$ in order to make the results insensitive to the precise form of $F^{(0)I}$.

[12]Our ν_n and $F^{(0)I}$ are defined in such a way as to make $F^{(0)I}$ incapable of oscillating between any two adjacent points. If we modify the forms of the $F^{(0)I}$ by

letting the points $\nu_i = 1, 2, 3$ assume the roles previously played by the points $\nu_i = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}$, then the δ_I change less than 1° below 500 MeV, while $\delta_0(730)$ and $\delta_2(730)$ change about 4° and 1°, respectively.

[13]Weinberg and Schwinger use different values for F_π which lead to $L = 0.115$ and 0.097, respectively. Weinberg's predictions for the π-N scattering lengths (Ref. 3) agree almost precisely with experiment if $L = 0.099$.

[14]S. Weinberg, private communication.

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[16]In addition to Eqs. (3), we can impose three second-derivative crossing relations on the S, P, and D waves. [Cf. Ref. 2, Eqs. (3.12), (3.13), and (3.14).] These relations may have implications for an isoscalar resonance. Furthermore, imposition of all six crossing relations might enable us to predict the S and P waves with no input except the D waves.

S-MATRIX CALCULATION VERSUS PERTURBATION CALCULATION IN QUANTUM ELECTRODYNAMICS*

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The use of the S-matrix dispersion-theoretic technique in the realm of quantum electrodynamics may be a matter of personal taste. Nevertheless, the demand for a consistent answer between the S-matrix calculation and the perturbation calculation, when both methods are supposed to be valid, is not entirely an idle question.

In an extensive review article,[1] Chou and Dresden (CD) succeeded in rederiving a number of well-known lower-order perturbation results of quantum electrodynamics by the use of the S-matrix technique. However, in one particular example, namely, the third-order vertex part with an off-shell external electron line,[2] the situation was somewhat unsatisfactory. The S-matrix calculation of CD for this example was incomplete. In order for their calculation to yield the same result as obtained from the conventional perturbation calculation by Akhiezer and Berestetskii (AB),[3] a conjecture on a certain integral identity was proposed by CD.

We wish to point out the following observation: (a) The conjecture of CD is not borne our by an explicit calculation. CD's final answer therefore does not coincide with the perturbation result of AB.

(b) The perturbation calculation has been repeated. Our result confirms that given by AB.[4]

(c) The discrepancy is traced. The way out of the quandary in the S-matrix approach is shown.

The problem involves the calculation of the amplitude corresponding to the third-order vertex part in electrodynamics where the external photon (momentum q) and one of the two external electrons (momentum p) are on the mass shells. The other electron (momentum $p+q$) is in general off-shell, except when the external photon has null four-momentum. The latter possibility corresponds to the subtraction term that is needed in performing the regularization.

The vertex part is given by

$$T(p+q,p,q) = \mathrm{const}\left(\frac{m}{2\omega_q E_p}\right)^{1/2} \bar{\psi}(p+q) e_\mu \Lambda_\mu(p+q,p,q) u(p), \tag{1}$$

$$\Lambda_\mu(p+q,p,q) = (\alpha/2\pi)[A\gamma_\mu + B(\gamma q)\gamma_\mu + Cp_\mu + D(\gamma q)p_\mu]. \tag{2}$$

The expressions for B, C, and D given by CD are identical to those obtained in the perturbation theory. These terms are stable, so to speak. The difference occurs in the coefficient of the γ_μ term in Eq. (2).

(i) In the perturbation calculation, we have

$$A^{\mathrm{pert}} = a^{\mathrm{pert}} + \frac{(\rho-2)}{2(\rho-1)} \ln\rho + \frac{1}{\rho}[F(\rho-1) - F(-1)], \tag{3}$$

$$a^{\mathrm{pert}} = \ln(\Lambda/m) + \tfrac{1}{4}, \tag{4}$$