[2]R. Klapisch, C. Philippe, J. Suchorzewska, C. Detraz, and R. Bernas, Phys. Rev. Letters 20, 740 (1968).

[3]At 1500°C on Ta, the efficiency of ionization taken from the Saha-Langmuir formula is 100 % for Cs and Rb while Ba and Sr have efficiencies, respectively, 1000 and 6000 times smaller. Thus, among the reaction products it is safe to consider this method as selective for Rb and Cs. Natural barium present in great quantities can, of course, be ionized (see text).

[4]Any contamination by a stable isotope will of course be included in measurement b. The difference between a and b is usually significant except in one case at masses 133 to 138 (see text).

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CHARGE DISTRIBUTION IN NUCLEI

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In recent years there has been great improvement in the accuracy of experiments on muonic x rays and electron scattering by nuclei. Analysis of these experiments by Ravenhall, Herman, and Clark[1] has yielded much improved knowledge on the charge distribution in nuclei. In particular it has been established that one way to fit the experiments is to use a charge distribution inside the nucleus of Pb^{208} that has a wine-bottle shape.

Ravenhall has already pointed out, and we wish to emphasize further, that such a detailed determination of the charge distribution, which yields some third shape parameter, is only possible by a combination of the experimental evidence from electron scattering and muonic x rays. These two experimental techniques measure different properties of the charge distribution. (In this statement, we exclude the scattering of low-energy electrons, of 20-50 MeV, which measures essentially the mean square radius of the charge, just like the mu x rays. These two pieces of evidence on $\langle r^2\rangle$ are in excellent agreement.)

The electron scattering at medium energy (150-750 MeV) measures essentially the value of r at which $\rho(r)$ has its steepest slope. This was shown by one of us (L.R.B.E.) in explicit calculations of the electron scattering from various assumed charge distributions.[2] He used the standard "Fermi" distribution, the distribution suggested by one of us (H.A.B.) on theoretical grounds, and a distribution calculated by himself from single-particle wave functions. The first two of these are

$$\rho_F/\rho_0 = \left[1+\exp\left(\frac{r-R_1}{a_1}\right)\right]^{-1} \quad \text{(Fermi)}, \tag{1}$$

$$\rho_B/\rho_0 = \left[1-\exp\left(\frac{r-R_2}{a_2}\right)\right]^{2} \quad \text{(Bethe)}. \tag{2}$$

When the parameters were adjusted for best fit to the experimental electron-scattering cross section, all three distributions gave essentially the same location of the position of steepest slope, viz. 6.5 F. The agreement is within about 0.03 F. Similar results were obtained by Lin[3] who did similar calculations for distributions (1) and (2), using the approximate method of Yennie, Boos, and Ravenhall.[4] The values of $\langle r^2\rangle^{1/2}$ derived from these various charge distributions differ quite appreciably, by up to 0.2 F.[5]

The empirical result that electron scattering measures the steepest slope of the charge distribution can be understood by using the Born approximation for scattering. We are of course well aware that the Born approximation is not valid for electron scattering by Pb. However, the approximation provides a heuristic guide and has been used for this purpose in the past. Especially the work of Yennie, Boos, and Ravenhall[4] has shown that modification of the Born approximation, taking into account the distortion of the electron wave functions by the Coulomb field, can give a good account of the scattering. In the Born approximation the scattered amplitude is given by

the Fourier transform of the charge density, thus

$$f(q)=q^{-1}\int\rho(r)(\sin qr)rdr. \quad (3)$$

In the most important experiments, q is of the order of 1 F^{-1}. For such large q it is convenient to integrate (3) by parts, obtaining

$$f(q)=q^{-3}\int(d\rho/dr)(\sin qr-qr\cos qr)dr. \quad (4)$$

This shows that the main contribution arises from the regions in which $d\rho/dr$ is large. This explains the result from the numerical analysis. [A further integration by parts of (4) is not useful.]

On the other hand, the energy levels of muonic atoms, in particular the 1s level, are determined essentially by $\langle r^2\rangle$. This has been demonstrated in the numerical work by one of us,[2] and will be further discussed analytically by the other.[6] The measurements of the K x rays therefore determine $\langle r^2\rangle$ very accurately. This quantity, however, is not uniquely related to the position of the steepest slope. As was already pointed out, the values of $\langle r^2\rangle^{1/2}$ from distributions (1) and (2) differ by about 0.2 F. Neither of them agrees with experiment; experiment requires[2] an $\langle r^2\rangle^{1/2}$ larger by 0.1-0.3 F.

The two pieces of evidence can be brought into agreement by changing the internal distribution of charge, keeping the position of the steepest slope unchanged. If we use a wine-bottle shape,

$$\frac{\rho_W}{\rho_0}=\left(1+w\frac{r^2}{R^2}\right)\frac{\tilde{\rho}}{\rho_0}, \quad (5)$$

where $\tilde{\rho}$ denotes either distribution (1) or (2), we can obtain, within wide limits, any value of $\langle r^2\rangle$, without appreciably shifting the position of the point of steepest slope. This fit with a parabolic shape has been used by Ravenhall, Herman, and Clark,[1] with excellent success. We have modified our distribution (2) as follows:

$$\rho_W/\rho_0=\left(1+w\frac{r^2}{R_3^2}\right)[1-\tfrac{1}{2}e^{(r-R_3)/a_2}]^{-2}, \quad r<R_3;$$

$$\rho_W/\rho_0=(1+w)\tfrac{1}{4}e^{-\gamma(r-R_3)}, \quad r>R_3;$$

$$R_3=R_2-a_2\ln 2. \quad (6)$$

We retained the values of R_2 and a_2 from our previous calculation,[3] viz. $R_2=7.70$ F, $a_2=1.20$ F. The parameter γ was deduced from the binding energy and taken to the 1.8 F^{-1}, while w was determined so as to give the correct value for $\langle r^2\rangle^{1/2}$, viz.[7] 5.50 F; this gave

$$w=0.717.$$

The resulting radial distribution is plotted in Fig. 1 (solid curve), and compared with that of Ravenhall, Herman, and Clark (dashed curve). The agreement is remarkably close, and might be further improved if the parameters in (6) were fitted by direct numerical calculation. Thus the parametrization (6) and that of Ravenhall will be difficult to tell apart experimentally. The maximum density, according to Fig. 1, is about 15% higher than the central density. From a rather crude application of the theory of nuclear matter, taking into account Coulomb and symmetry energy, one of us[8] derived a density distribution for Pb which showed a maximum about 10% higher than the central density. The agreement is very satisfactory; the theory will be refined.

Also on Fig. 1, we have plotted the density distribution based on single-particle wave functions in a Saxon-Woods well, which gives a best fit to the electron-scattering results[2] (dot-dash curve). This agrees very well indeed with the other distributions in the region of greatest slope, but it is significantly larger near the center of the nucleus. In this distribution, the Coulomb repulsion is of course included, and the central density would be even higher if it

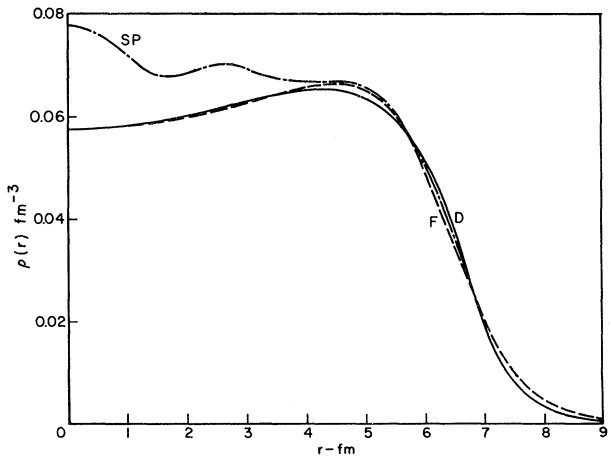

FIG. 1. Radial distributions of charge in Pb, fitting electron scattering. F = Fermi-type distribution, Eq. (1); D = differential Thomas-Fermi theory, Eq. (2). Both of these have been multiplied by a wine-bottle factor, Eq. (5), and both fit also the muonic x rays. SP = single-particle wave functions, fitting electron scattering but not muonic x rays.

were not. Thus the single-particle distribution does not give a central dip in the distribution. The resulting value of $\langle r^2 \rangle^{1/2}$ is therefore too small[2] by about 0.1 F. It is possible that the use of a nonlocal central potential, which tends to lower the wave functions in the central region and increase them in the outer regions,[9] may reconcile this difference. Calculations along these lines are in progress.

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