in high fields should restore R to its normal value. Thus, a decrease of $|R|$ with B is predicted by the CDW model. The magnitude of the decrease should vary from specimen to specimen, depending on the orientation of $\vec{Q}$ relative to the crystallographic axes. The large decrease observed in the specimen of Fig. 1 would require f to be as large as 5%. This sample had a magnetoresistance of ~600% at 110 kG, which indicates[7] independently that f was likely this large.

Blount[8] has shown that the probability of magnetic breakdown at an energy gap Δ depends on the parameter $(\hbar\omega_c E_F)^{1/2}/\Delta$, where E_F is the Fermi energy. Quantitative explanation of the magnetoresistance required heterodyne gaps with $\Delta \sim 0.07$ eV.[7] Accordingly, the foregoing expression is about 0.3 at 50 kG. We conclude that the CDW model provides a consistent explanation of both Hall effect and magnetoresistance anomalies in K.

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DIVERGENCES AND PHONONS IN THE INDIRECT INTERACTION*

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Divergences arising in the third-order perturbation-theory derivation of the indirect interaction between nuclear or ionic spins in a metal are avoided if the recoil of the localized spin is taken into account by including the phonon degrees of freedom. When this is done the form of the usual Ruderman-Kittel function is altered, particularly at small R, where the singularity at the origin is removed.

A recent paper by Vertogen and Caspers[1] extended the perturbation-theory calculation of the indirect interaction between ionic or nuclear spins in a free-electron gas to third order in the exchange interaction J. The third-order contribution to the indirect interaction was divergent, bringing into question the validity of the RKKY interaction[2] and the perturbation methods of its derivation.

In this Letter it will be argued that the VC divergences have a well understood physical basis, and that they arise as an artifact of the usual model system for the contact interaction between a localized spin $\vec{I}_l$ at $\vec{R}_l$ and an electron of spin $\vec{S}_i$ at $\vec{r}_i$:

$$V = J\sum_{li} \vec{I}_l \cdot \vec{S}_i \delta(\vec{R}_l - \vec{r}_i). \tag{1}$$

The choice of a physically more realistic model for the contact interaction eliminates the VC divergences, as well as a singularity in the RKKY function.[3]

The usual model for the contact interaction between a localized spin and a sea of electrons assigns the localized spin's nucleus a definite equilibrium position at which it remains fixed. In the usual Born representation the potential term in the Hamiltonian represents scattering of the electrons from a fixed scatterer. The Hamiltonian in this representation is

$$H = \sum_{\vec{k}s} \epsilon_{\vec{k}s} c_{\vec{k}s}^{\dagger} c_{\vec{k}s} + \frac{J}{\Omega} \sum_{l\vec{k}\vec{k}'ss'} \times \exp[i(\vec{k}-\vec{k}')\cdot\vec{R}_l] \vec{I}_l \cdot \vec{\sigma}_{ss'} c_{\vec{k}s}^{\dagger} c_{\vec{k}s}, \tag{2}$$

where the $c_{\vec{k}s}^{\dagger}$, $c_{\vec{k}s}$ are creation and annihilation operators for electron states of energy $\epsilon_{\vec{k}s}$, J is the coupling, $\vec{I}_l$ is the spin operator of the spin localized at $\vec{R}_l$, and $\sigma_{ss'}$ are the matrix elements of the Pauli spin matrices.

The second term in Eq. (2) represents a scat-

tering amplitude which is the same for all momentum transfers modulo a phase factor. This implies that the probability of large momentum transfers is the same as for small momentum transfers. In particular, in weighted sums over intermediate states large momentum transfers would receive the same weight as small transfers. Examination of the divergent integrals in VC[4] reveals that their divergences are due to this equal weighting of large momentum transfers and small momentum transfers in the sums over intermediate states.

A more realistic model for the contact interaction must allow for recoil of the localized spin in a collision with the electrons. Since the localized spin is embedded in a solid, recoil generates phonons, and the probability of a scattering at a given momentum transfer is affected by the phonons. The lowest order consideration of this effect is to replace the Hamiltonian, Eq. (2), with an effective Hamiltonian which has been averaged over the displacements of the localized spin's atom from its equilibrium position. Such an averaging gives rise to a Debye-Waller factor in the scattering of each pair of plane-wave states. The resulting effective Hamiltonian is

$$H=\sum_{\vec{k}s}\epsilon_{\vec{k}s}c_{\vec{k}s}^{\dagger}c_{\vec{k}s}+\frac{J}{\Omega}\sum_{l\vec{k}\vec{k}'ss'}\exp[i(\vec{k}-\vec{k}')\cdot\vec{R}_l]\times\exp[-\tfrac{1}{2}\gamma^2(\vec{k}-\vec{k}')^2]\vec{I}_l\cdot\vec{\sigma}_{ss'}c_{\vec{k}s}^{\dagger}c_{\vec{k}'s'}, \quad (3)$$

where γ is the rms displacement of the localized spin's nucleus from its equilibrium position in the solid. When this effective interaction is used to determine the third-order contributions to the indirect interactions, the VC divergences are not present.

Barring a numerical computation in each case of interest, a comparison of the relative size of the second- and third-order contributions can be made using a criterion from the VC paper itself. If there is some cutoff wave vector $k_{c.o.}$ which restricts the size of the momentum transfer to low values, then third-order contributions are smaller than second order if $k_{c.o.}<\pi^2/m^*J$, where m^* is the electron effective mass. The phonons introduce an effective cutoff $k_{c.o.}=1/\gamma$ and the comparison of these two quantities in Table I shows that for the hyperfine interaction, second order may be expected to be exact. However, for s-d, s-f interactions, though the second order is still dominant, third order is somewhat comparable.

Table I. Comparison of $1/\gamma$ (Debye model) and π^2/m^*J.

Material and interaction	$1/\gamma$	π^2/m^*J
Ag hyperfine interaction[a]	$\sim 9k_F$	$\sim 10^7 k_F$
Gd s-f interaction[b]	$\sim 10k_F$	$\sim 38k_F$
CuMu s-d interaction[c]	$\sim 5k_F$	$\sim 25k_F$

[a]Ag parameters taken from Ref. 1.

[b]Gd parameters from T. Kasuya, in Magnetism, edited by G. T. Rado and H. Suhl (Academic Press, Inc., New York, 1966), Vol. IIB; K. A. Gschneidner, Rare Earth Alloys (D. Van Nostrand and Company, Inc. Princeton, N. J., 1961).

[c]CuMn parameters from Ref. 3 and C. Kittel, Introduction To Solid State Physics (John Wiley & Sons, Inc., New York, 1966).

The range dependence of the indirect interaction for second order is determined for the usual model by the RKKY function F, which is

$$F(x)=\frac{x\cos x-\sin x}{x^4}. \quad (4)$$

The phonon-modulated form F_γ depends on γ and can be expressed as a Fourier integral:

$$F_\gamma(\beta)=-\frac{1}{2\pi}\frac{\pi^{1/2}}{2\alpha^3}e^{-\frac{1}{4}(\beta/\alpha)^2}+\int_0^\infty dx\,\frac{\sin\beta x}{\beta}e^{-\alpha^2x^2}(1-x^2)\ln\left|\frac{1+x}{1-x}\right|, \quad (5)$$

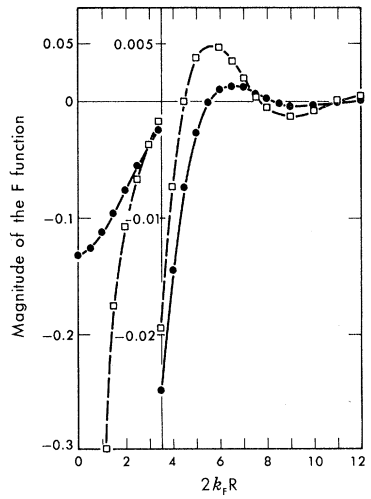

FIG. 1. A plot of the Ruderman-Kittel function F (circles) for $2k_F\gamma=0$ and the phonon-modulated F_γ (squares) for $2k_F\gamma=1$. Note the change of scale at $2k_FR=3.5$.

where $\beta = 2k_F R$ and $\alpha = 2k_F\gamma$.

At large R, F_γ and F do not differ greatly for small γ. However, for small R, F_γ achieves a finite value while F has a singularity at $R = 0$. The consequences of the differences between F_γ and F, and other effects of phonons on the model, will be discussed elsewhere.[5]

Figure 1 shows a comparison of F_γ and F for small R and the values $2k_F\gamma = 1$ and $2k_F\gamma = 0$.

As a final point, the relationship between the VC divergences and the Kondo singularity[6] in the resistivity should be clarified. Calculation of the Kondo terms in resistivity with the effective Hamiltonian in Eq. (3) alters Kondo's results only by a multiplicative factor

$$2\{1-\exp[-\tfrac{1}{2}(2k_F\gamma)^2]\}/(2k_F\gamma)^2$$

which approaches 1 as $\gamma \to 0$. The VC divergences arose from large momentum transfers while Kondo's singularity involves small energy and momentum transfers near the Fermi surface.

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RESONANCE FREQUENCIES OF THE ORTHOFERRITES IN THE SPIN REORIENTATION REGION*

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The purpose of this Letter is to present a theory of the resonance modes of a canted, orthorhombic antiferromagnet in the temperature region where the a-c plane anisotropy changes sign. The results are applied to the rare-earth orthoferrites in which the direction of the weak moment changes from the a axis to the c axis with increasing temperature.[1-4] It is found that low-frequency antiferromagnetic resonance modes occur in the reorientation region.

The theoretical results provide an interpretation for recent microwave absorption measurements[5] which have not been adequately explained. The results show that resonant absorptions should be observed at zero applied field for a wide range of frequencies. Additional results of the theory, also verified experimentally, include the fields for resonance along the a and c axes, the directions of maximum rf susceptibility, and the angular dependence of the resonance aplitude.

A two-sublattice model has been used to calculate the equilibrium position of the magnetization vectors and the resonance frequencies. The model is similar to the one used by Herrmann,[6] but has been extended to include the effects of fourfold anisotropy terms which are important in the region considered. A preliminary analysis[5] suggests that the relatively low resonance frequencies implied by Herrmann's results are eliminated when fourfold terms are included. However, it is shown here that if the fourfold anisotropy is consistently taken into account in the expressions for the equilibrium configurations and the resonance frequencies, then low-frequency modes are predicted by this model.

The energy expression for the model contains the following terms:

$$E = MH_E(\vec{m}_1\cdot\vec{m}_2) + M\vec{H}_D\cdot(\vec{m}_1\times\vec{m}_2) - M\vec{H}_0\cdot(\vec{m}_1+\vec{m}_2) + E_A. \quad (1)$$

Here, $\vec{m}_1$ and $\vec{m}_2$ are unit vectors in the directions of the sublattice moments, M is the magnitude of the moments, H_E is the isotropic exchange field, and H_0 is the applied field. The antisymmetric exchange field $\vec{H}_D$ is directed along the b axis. The anisotropy energy