NO-GO THEOREM*

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We show that in the covariant approach there exist no systems of single-particle functions satisfying a completeness property which generate any acceptable single-particle representations of current algebras and superconvergence relations. The nonexistence of any acceptable local infinite-component field theory or wave equation is also demonstrated.

The large number of sum rules derived from current algebra and superconvergence assumptions along with the distinctively bumpy structure of relevant cross sections have generated a strong interest in the possibility of finding nontrivial single-particle representations of classes of such sum rules.[1,2] It is well known that one must treat an infinite number of particles of arbitrarily high spin to obtain anything beyond trivial results,[3] and the many benefits of finding such a representation have been well described.[4] Unfortunately, the only known solutions have infinitely degenerate mass shells,[1,4,5] and give amplitudes and form factors with structure wildly different from what one would expect from normal crossing and analyticity.[6] Almost all of the attempts at finding a more interesting solution have been perturbative in nature; consistency could be checked only to finite order in the mass splitting.[1,2] Thus a direct "covariant approach" using a completeness property of the single-particle functions seemed more promising.[1,5]

On the other hand, there are those who have been searching for a local infinite-component field theory capable of describing the hadrons in the same spirit as the method of quasiparticles in nonrelativistic many-body theory.[7] Again the only known solutions satisfying the spectral condition have infinite degeneracy[8] and give amplitudes with wild structure.[6,9] These two approaches are intimately related; indeed, a single-particle saturation of a class of sum rules from the covariant approach appears to be simply equivalent or closely connected to the Born approximation of an infinite-component field theory.[6,9]

Since the existence of unsubtracted current-algebra sum rules presupposes certain superconvergence requirements,[10,11] we will consider only simultaneous single-particle representations of both. The idea of the "covariant approach"[1] is to find a set of c-number functions,

$$u_\alpha(\vec{p}\lambda[sn]),\ \bar{u}_\alpha(\vec{p}\lambda[sn]),\ v_\alpha(\vec{p}\lambda[sn]),\ \text{and}\ \bar{v}_\alpha(\vec{p}\lambda[sn]),$$

satisfying the conditions listed below. Here $\vec{p}$ denotes the three-momentum, λ the helicity, s the spin, and n the mass and any internal quantum numbers. By u we mean the functions describing mass shells in the upper light cone, and by v the lower.[12] The subscript α indicates a basis for any fully reducible representation of the complete Lorentz group [SL(2C) and spatial reflections].[13] We will show that no such functions exist satisfying the following conditions.

(I) Covariance:

$$\begin{Bmatrix} u_\alpha(\vec{p}\lambda sn) \\ v_\alpha(-\vec{p}\lambda sn)\end{Bmatrix} = \sum_{\alpha',\lambda'} [D(\Lambda^{-1})]_{\alpha\alpha'}[R^s(\Lambda,\vec{p},n)]_{\lambda\lambda'}\begin{Bmatrix} u_{\alpha'}(\Lambda\vec{p}\lambda' sn) \\ v_{\alpha'}(-\Lambda\vec{p}\lambda' sn)\end{Bmatrix}, \tag{1a}$$

$$\begin{Bmatrix} \bar{u}_\alpha(\vec{p}\lambda sn) \\ \bar{v}_\alpha(-\vec{p}\lambda sn)\end{Bmatrix} = \sum_{\alpha',\lambda'} [D^{-1}(\Lambda^{-1})]_{\alpha'\alpha}[R^{s\dagger}(\Lambda,\vec{p},n)]_{\lambda\lambda'}\begin{Bmatrix} \bar{u}_{\alpha'}(\Lambda\vec{p}\lambda' sn) \\ \bar{v}_{\alpha'}(-\Lambda\vec{p}\lambda' sn)\end{Bmatrix}, \tag{1b}$$

where $\Lambda \in \mathrm{SL}(2C)$, $D(\Lambda)$ is the representation of SL(2C) just described, and R^S is the Wigner rotation.

(II) Reasonable mass spectrum: The possible values of p^2 described by the functions contain some (at least one!) mass shells with finite degeneracy (and obviously only timelike values can occur).

(III) Reasonable form factors: $D(\Lambda)$ contains at least one infinite-dimensional irreducible representation[14] and possesses a four-vector operator[15] Γ_μ;

$$D(\Lambda)\Gamma_\mu D^{-1}(\Lambda) = \Lambda_{\mu\nu}\Gamma_\nu.$$

(IV) Completeness property[12]:

$$\sum_{n,s,\lambda,\alpha} u_\alpha(\vec{p}\lambda ns)\bar{u}_{\alpha'}(\vec{p}\lambda ns)(\Gamma_0)_{\alpha'\beta}(\vec{p}^2+m_n^{\,2})^{-1/2} \pm \sum_{n,s,\lambda,\alpha'} v_\alpha(-\vec{p}\lambda ns)\bar{v}_{\alpha'}(-\vec{p}\lambda ns)(\Gamma_0)_{\alpha'\beta}(\vec{p}^2+m_n^{\,2})^{-1/2} = \delta_{\alpha\beta}, \tag{2}$$

where the positive-energy projection operator [first term in Eq. (2)] is bounded by a polynomial in $\vec{p}$. The point here is that for current algebra we can tolerate negative-energy functions since as $p_z \to \infty$, the positive-energy functions decouple from the negative-energy ones (z diagrams vanish),[1,4] but for superconvergence relations we must have the positive-energy projection operator bounded by a polynomial.[16]

Observe that condition (III) and footnote 14 imply that $D(\Lambda)$ must be a representation admitting a nondegenerate Hermitian bilinear form (if we are to ever contemplate scalar currents):

$$\bar{u}_\alpha(\vec{p}\lambda sn) = \sum_\beta u_\beta^*(\vec{p}\lambda sn) g_{\beta\alpha}, \tag{3}$$

where g is a nondegenerate Hermitian operator (clearly $g\Gamma_0$ must be Hermitian). For if g were not Hermitian the charges would not be real, and if g were degenerate there would exist at least one vector in a basis for $D(\Lambda)$ orthogonal to all vectors in that basis not identically zero.[17]

No-go theorem.—A system of functions satisfying conditions (I) through (IV) above cannot exist.

The proof of this theorem rests upon a remarkably restrictive property of classes of functions over Minkowski spaces first proved for functions of one four-vector ten years ago by Bogoliubov and Vladimirov.[18] For functions of many four-vectors, this property was recently proved by Bros, Epstein, and Glaser.[19] We need the result only for functions of one four-vector.

Theorem of finite covariance.—Let $F(x_\mu)$ $[G(x_\mu)]$ be any tempered distribution over x_μ whose spectral decomposition contains only timelike and lightlike four-momenta in the upper (lower) cone. Further suppose that $F(x_\mu) = G(x_\mu)$ when $x^2 < 0$. Then $F(x_\mu)$ and $G(x_\mu)$ are finite covariants: $F(x_\mu)$ transforms under SL(2C) as a finite sum of finite-rank tensors, and indeed,

$$F(x_\mu) = \sum_{i=1}^{L} Q_i(x_\mu) F^i(x_\mu), \tag{4}$$

where the Q_i are finite polynomials and the F^i are invariants, and similarly for $G(x_\mu)$.

Proof of the no-go theorem.—Invent the free field

$$\psi_\alpha(x) = \sum_{n,s,\lambda} \frac{d^3p}{(\vec{p}^2+m_n^{\,2})^{1/2}} \exp\{-i\vec{p}\cdot\vec{x} + i(\vec{p}^2+m_n^{\,2})^{1/2}t\} u_\alpha(\vec{p}\lambda sn) a(\vec{p}\lambda sn)$$
$$+ \sum_{n,s,\lambda} \frac{d^3p}{(\vec{p}^2+m_n^{\,2})^{1/2}} \exp\{i\vec{p}\circ\vec{x} - i(\vec{p}^2+m_n^{\,2})^{1/2}t\} v_\alpha(\vec{p}\lambda sn) b^\dagger(\vec{p}\lambda sn), \tag{5}$$

where $a^\dagger$ and $b^\dagger$ are the usual creation operators satisfying canonical anticommutation or commutation relations depending upon the sign in Eq. (2). Consider the two-point functions

$$F_{\alpha\beta}(\eta_\mu) = \langle\psi_\beta(y_\mu)|\psi_\alpha(x_\mu)\rangle$$

and

$$G_{\alpha\beta}(\eta_\mu) = \mp\langle\psi_\alpha^\dagger(x_\mu)|\psi_\beta^\dagger(y_\mu)\rangle, \tag{6}$$

where $\eta_\mu = (y-x)_\mu$; and again the sign depends upon the sign in Eq. (2). It is easy to show that F and G are tempered distributions since the positive- and negative-frequency projection operators [the separate terms in Eq. (2)] are polynomial bounded. Form the (anti-)commutator function

$$C_{\alpha\beta}(\eta_\mu) = F_{\alpha\beta}(\eta_\mu) - G_{\alpha\beta}(\eta_\mu). \tag{7}$$

It follows immediately from Eq. (2) in the usual way that

$$C_{\alpha\beta}(\vec{\eta}, 0) = X_{\alpha\beta}\delta^3(\vec{\eta}), \tag{8}$$

where $X_{\alpha\beta}$ depends upon Γ_0 and g,[20] and from covariance [condition (I)],

$$C_{\alpha\beta}(\eta_\mu) = 0, \quad \eta^2 < 0 \text{ (locality)}. \tag{9}$$

Thus, F and G satisfy all of the conditions of the theorem of finite covariance: F and G are finite covariants.

The covariance condition (I) then tells us that

$$G_{\alpha\beta}(\eta_\mu) = \sum_{i=1}^{L}\sum_{\alpha'\beta'} g_{\alpha\alpha'} Q_{\alpha'\beta'}{}^{i}(\eta_\mu) \times G_{\beta'\beta}{}^{i}(\eta_\mu), \tag{10}$$

where the $Q_{\alpha\beta}{}^{i}(\eta_\mu)$ are finite polynomials of finite-rank tensor operators in $D(\Lambda)$ and the $G_{\alpha\beta}{}^{i}(\eta_\mu)$ are invariant operators in $D(\Lambda)$. But all finite-rank tensor operators can only connect a finite number of vectors in a canonical basis for $D(\Lambda)$ and g is semidiagonal,[17] so that for any fixed β there must be an infinite number of α's such that

$$G_{\alpha\beta}(\eta_\mu) = \langle\psi_\alpha^\dagger(x_\mu)|\psi_\beta^\dagger(y_\mu)\rangle = 0$$

for all η_μ. Thus given the state $|\psi_\beta^\dagger(x_\mu)\rangle$, there are an infinity of states $|\psi_\alpha^\dagger(y_\mu)\rangle$ orthogonal to it, independent of x_μ and y_μ. Smear the states with a test function in $\varphi^4(x,y,z,t)$ whose Fourier transform is of compact support containing only one mass shell in the spectrum of the u's;

$$|\psi_\beta^\dagger(f_m)\rangle = \int d^4x f_m(x_\mu)|\psi_\beta^\dagger(x_\mu)\rangle. \tag{11}$$

Then $\langle\psi_\alpha^\dagger(f_m)|\psi_\beta^\dagger(f_m)\rangle = 0$ for any fixed β and an infinite number of values of α and for any wave packet on the mass shell m: There must be an infinite degeneracy on each mass shell. This of course violates condition (II).

We have also proved the Theorem: The set of acceptable local infinite-component field theories is void. Thus all possible theories either are just stacks of the well-known mass-degenerate ones or else violate the spectral condition (asymptotic spacelike modes will exist). Concerning invariant equations, we have proved the Theorem: Any wave equation defined in any representation of $D(\Lambda)$ described above which can be written as an eigenvalue equation for an energy operator at fixed $\vec{p}$, Hermitian under some positive definite metric, and which possesses at least one timelike solution of finite degeneracy, either decouples into a finite set of finite-dimensional equations or else must possess spacelike solutions.[21]

It is clear that this result would not be obvious in a perturbative calculation around some degenerate timelike mass, since the spacelike solutions would only appear when all orders in the power series were taken into account. Also it is clear that going to a "unitary-spin" explosion model will not help; the result holds for any reasonable representation of SL(2, C), whether irreducible, or from SU(6, 6) or SU(136, 136).

Once again the structure of the Poincaré group prevents us from extending simple intuitive nonrelativistic models into the relativistic regime; the Galilean group allows us to "separate" internal structure from external interactions, but this apparently cannot be done relativistically. We must deal with poles and cuts.

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[12]Although for simplicity we use the same labels, we do not mean to imply that the positive-energy functions must be the same as the negative-energy functions (up to the sign of the energy): *PCT* does not concern us here [E. Abers, I. T. Grodsky, and R. E. Norton, Phys. Rev. 159, 1222 (1967)].

[13]This is not a real restriction since one could not diagonalize the particle parameters if the representation were not fully reducible. We pretend that α is discrete in the text; it can of course be continuous.

[14]This is the spirit of the game; we want the "Born approximation" to the form factors, $\bar{u}(\vec{p}'\lambda'n's')\Gamma_0 u(\vec{p}\lambda ns)$, to be more realistic than in the simple quark model using ordinary fields; for example, it should have nonkinematical singularities in $t=[p(n)-p'(n')]^2$.

[15]If Γ_μ is assumed to be p dependent, we have nothing to say, except that then there is no known technique for finding sets of u's satisfying condition (IV).

[16]More generally, we must have (symbolically)

$$\sum_n u(\vec{p}n)\bar{u}(\vec{p}n)\Gamma_0/\mathcal{E}_n + X(\vec{p}) = 1,$$

where as $\vec{p}\to\infty$, $\bar{u}\Gamma_0 Xu\to 0$. It is easy to show that $X(\vec{p})$ cannot be identically zero, and the choice in the text is clearly the most obvious one.

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[20]We are assuming that all matrices involved possess inverses, but this is not crucial to the proof.

[21]In particular, an equation of the form $(\gamma\cdot p-\mathfrak{M})\psi=0$, where γ_μ is the Dirac four-vector and ψ transforms under the direct product of the Dirac and any unitary representation ($\gamma_0\mathfrak{M}$ Hermitian, of course), must have spacelike solutions unless $\mathfrak{M}=m_0$, a number. This result was conjectured by Abers, Grodsky, and Norton, Ref. 12, and apparently contradicts a conclusion of Ref. 1. (We see no reason why the positive-energy spacelike solutions should decouple from all the positive-energy timelike solutions as $p_z\to\infty$.)

DIRECT PRODUCTION OF MUONS IN COSMIC RAYS?

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In a recent paper Bergeson et al.[1] have described measurements of the intensity of cosmic-ray muons as a function of zenith angle and depth underground. The results are of considerable interest in that they show almost no dependence of intensity on zenith angle, in strong contradiction to the secθ enhancement expected if the muons derive from pions and kaons. Bergeson et al. conclude that the majority of cosmic-ray muons of energy above 1000 GeV are produced either directly or as decay products of very short-lived secondaries ($\tau \ll 10^{-8}$ sec), and their published results indicate that direct production is dominant as low as 500 GeV.

The purpose of the present work is to exam-