MESON BOOTSTRAP WITH FINITE-ENERGY SUM RULES*

Christoph Schmid
Lawrence Radiation Laboratory, University of California, Berkeley, California
(Received 2 January 1968)

Finite-energy sum rules (FESR)[1,2] express analyticity and tie together the high-energy and the low-energy behavior of scattering amplitudes. Assuming that the high-energy amplitude is dominated by a few Regge poles in the crossed channel (t), and that the low-energy amplitude is dominated by a few direct-channel (s) resonances, the FESR allow us to determine the t-channel Regge parameters in terms of the parameters of the s-channel resonances. In the $\pi\pi$ system both the s and t channels contain the same particles, therefore we obtain self-consistency–or bootstrap–conditions.[3] We show how resonances in the direct $\pi\pi$ channel (ρ, f, g) generate (via FESR) the ρ Regge pole in the t channel, and we calculate $\alpha_\rho(t)$ for α from $-\frac{1}{2}$ to $+3$. This is relevant to the intriguing question of elementarity versus compositeness of particles: If we assume that the $I_t=1$ amplitude is dominated by one Regge pole, the ρ, then our model predicts that this pole is moving, with $d\alpha/dt \approx 1.0$ GeV^{-2}, and it cannot be a fixed Regge pole (elementary particle).[4] We also treat the superconvergent $I_t=2$ and the $I_t=0$ amplitudes (with P and P'), and we solve the ρ-f bootstrap system.

The Regge amplitude at fixed momentum transfer t for high energies s is

$$A \approx \beta\left(\frac{\nu}{\nu_0}\right)^{\alpha} \frac{\mp 1 - e^{-i\pi\alpha}}{\sin \pi\alpha} \text{ for } I_t = \begin{Bmatrix} 0,2 \\ 1 \end{Bmatrix}, \tag{1}$$

where $\nu \equiv \frac{1}{2}(s-u) = \frac{1}{2}z_t(t-4m_\pi^2)$, $\nu_0 \equiv 1$ BeV^2, and β is the reduced residue function (regular at threshold). If the asymptotic formula (1) is good for $\nu > N$, then the following FESR are equally good[1]:

$$\int_0^N d\nu \nu^n \operatorname{Im}A(\nu,t) \equiv S_n(N,t) \approx \beta(t)\frac{N^{\alpha(t)+n+1}}{\alpha(t)+n+1}. \tag{2}$$

For the low-energy (LE) integral on the left-hand side (LHS) of (2) we use the narrow-resonance approximation:

$$\text{LHS} = C_{ts}[s/(s-4m_\pi^2)]^{\frac{1}{2}} 2\pi(2l+1)[m\Gamma x]P_l(z_s)\nu^n, \tag{3}$$

where C_{ts} is the isospin crossing matrix, m is the resonance mass, Γ its width, and x its elasticity.

There are two approaches to any bootstrap, old[5] or new[3]: (a) One uses the physical masses and couplings in the s channel as input on the LHS to compute the corresponding output information in the t channel on the right-hand side (RHS) of (2). One then asks whether the input and output parameters are consistent, $m_{\text{in}}^{(s)} = m_{\text{phys}} \overset{?}{\simeq} m_{\text{out}}^{(t)}$. (b) One solves the system of bootstrap equations requiring $m_{\text{in}} = m_{\text{out}}$, and then one checks that the self-consistent parameters are approximately equal to the physical ones, $m_{\text{in}}^{(s)} = m_{\text{out}}^{(t)} \overset{?}{\simeq} m_{\text{phys}}$. In approach (a) we test consistency. In approach (b) we mix up consistency and stability, and we also test uniqueness. We shall mostly use (a), since it is much easier for computations.

We work at fixed t and with definite isospin, I_t, in the t channel. We start with $I_t=1$. The RHS of the FESR is therefore given by the ρ Regge term. This amplitude is odd in z_t, therefore we can use S_0, S_2, etc. We work at $t=m_\rho^2$ and not at $t=0$, because we know β only at $t=m_\rho^2$. At $t=m_{\text{res}}^2$ we have

$$\beta(t=m_{\text{res}}^2) = (m\Gamma x)(d\alpha/dt)\pi(2l+1)c_l[2\nu_0/(t-4m_\pi^2)]^l[t/(t-4m_\pi^2)]^{\frac{1}{2}}, \tag{4}$$

where c_l is the leading coefficient of $P_l(z)$. $\beta(t=0)$ is unknown. The approximation[6] $\beta(t) \approx \beta(0)$ can not be used, because it is undefined unless we specify the value of ν_0. If we change ν_0, β will pick up an exponential t dependence. Choosing $\nu_0 = 1$ BeV^2 we obtain the output $\beta(m_\rho^2)/\beta(0) = 3.0 \pm 0.1$.

By far the most important input resonances[7] are the ρ, $f(1250)$, and $g(1650)$. Therefore we consider the following three cases: Limit of integration N is (I) above the ρ, (II) above the f, and (III) above the g. We choose N halfway between the highest resonance included and the one immediately above. A reasonable range for N about the halfway point is $\delta N=\pm 0.15$ BeV^2, as explained below. (Alternatively we could allow N to vary from the midpoint between the two resonances half the remaining distance to the next resonance, i.e., by $\delta N \approx \pm 0.25$ BeV^2.)

In case (I) we take only the ρ on the LHS of (2); we use the experimental value $d\alpha/dt=1$ BeV^{-2} for connecting β and Γ [Eq. (4)], and from S_0 at $t=m_\rho^2$ we obtain $\Gamma_\rho^{\rm out}/\Gamma_\rho^{\rm in}=0.95\pm 0.21$, where $\Gamma_\rho^{\rm out}=\Gamma_\rho^{t}$ and $\Gamma_\rho^{\rm in}=\Gamma_\rho^{s}$. This should be compared with the value in the old bootstrap[5] $\Gamma^{\rm out}/\Gamma^{\rm in}\sim 5$-10. Our result depends crucially on the value of the crossing matrix element $C_{11}=\frac{1}{2}$ and on the ρ spin. It also depends on N, and the uncertainty $\delta N=0.15$ BeV^2 produces the error in this and all following results. Because the FESR are linear in the amplitudes, we can compute only the ratio $\Gamma^{\rm out}/\Gamma^{\rm in}$, while the absolute value of Γ_ρ drops out of the equations. Γ_ρ (the ρ-coupling constant) merely serves to fix the scale of all amplitudes.

In case (II) we use the ρ and the f as input on the LHS and from $S_0(t=m_\rho^2)$ we obtain $\Gamma_\rho^{\rm out}/\Gamma_\rho^{\rm in}=0.84\pm 0.11$, where the error refers to δN only. There are various ways of re-expressing this result. For example, we can require self-consistency, assume that m_ρ and m_f are given, and compute $\Gamma_f/\Gamma_\rho=1.01\pm 0.18$. The experimental ρ width is not well known: Rosenfeld[7] gives for the experimental ratio 0.91. Alternatively we can require exact self-consistency, take the experimental widths, and use S_0 to determine the cutoff N. We obtain $s_N\equiv s(\nu=N)=1.92$, which should be compared with the half-way point $s_N=2.12$.

Since we now have a broader support we can also use the higher moment sum rule S_2. From S_0 and S_2 we determine the output $\alpha(t)$:

$$S_2/(N^2S_0)=(\alpha+1)/(\alpha+3),$$

$$\alpha=(3S_2-N^2S_0)/(N^2S_0-S_2). \qquad (5)$$

Using $s_N=1.92$, determined above from S_0, we obtain $\alpha(m_\rho^2)=1.1\pm 0.4$ and $\alpha(0)=0.4\pm 0.3$.

Next we treat case (III) with (ρ, f, g) as input. g has an unknown 2π branching ratio[7] x_g. We impose $\Gamma_\rho^{\rm out}=\Gamma_\rho^{\rm in}$ and use S_0 to determine x_g; we get $x_g=58\pm 8\,\%$. If we use S_2 we have $x_g=58\pm 12\,\%$. Combining S_0 and S_2 to eliminate β_ρ, we obtain $\alpha_\rho(m_\rho^2)=1.0\pm 0.3$ and $\alpha_\rho(m_g^2)=2.9\pm 0.8$.

Next we note that the $P_l(z)$'s in the amplitudes corresponding to the three input resonances ρ, f, and g all have their first zeros simultaneously at $t\approx -0.3$ BeV^2, or more precisely at -0.26, -0.31, and -0.29, respectively. Therefore the RHS will vanish near this point: $\beta_\rho(-0.3)\approx 0$. Inclusion of the low partial waves neglected on the LHS will shift this value downward by $\delta t\approx 0.1$ BeV^2. Let us check whether this zero of $\beta_\rho(t)$ is connected with the vanishing of $\alpha_\rho(t)$. Unfortunately (5) gives $\alpha=0/0$ if $\beta=0$. Therefore we go to $t=-0.75$ BeV^2 and check if α becomes negative. We obtain $\alpha_\rho(-0.75)=-0.4\pm 0.1$. Interpolation between the t values gives $\alpha_\rho=0$ for $t\approx -0.3$ BeV^2. Summarizing, an input of ρ, f, and g in the s channel is able to generate an output $\alpha_\rho(t)$, with $-\frac{1}{2}\leq\alpha_\rho\leq +3$. Our model predicts that the Regge pole in the crossed channel must be a moving pole with $d\alpha_\rho/dt=1.0\pm 0.2$ BeV^{-2}.

If there were only one ρ Regge pole, then factorization would lead to a contradiction between the one-ρ-pole approach to πN charge exchange, and $\pi\pi$ elastic scattering. In the former case[3] only the helicity flip amplitude, $\beta_{\rm cex}$, vanishes for $\alpha_\rho(t)=0$, while the nonflip amplitude $A_{\rm cex}'$ remains nonzero for $\alpha=0$. In a one-ρ-pole approach this indicates that the ρ trajectory chooses sense at $\alpha=0$, while the present analysis indicates that ρ chooses nonsense. This contradiction disappears if we assume we have one effective ρ trajectory, which simulates the combined effect of ρ and ρ'.[8]

The $I_t=2$ amplitude does not contain any known Regge pole, and is therefore superconvergent. Since it is even in z_t we can test the relation $S_1=0$. The contributions of ρ, f, and g at $t=0$ are -0.34, $+1.24$, and -4.10, and have the tendency to cancel on the LHS, thus producing a zero on the RHS of (2). On the other hand, the convergence of the LHS is bad, the magnitudes increasing like $\nu^{3/2}$ in S_1. This is because the sole difference between the ρ-producing sum rules for $I_t=1$ and the superconvergent sum rules for $I_t=2$ is a sign in the crossing matrix. The only way to simultaneously generate the ρ in $I_t=1$ and superconvergence in $I_t=2$ is to have very large, but strongly overlapping, resonances, or nonresonating contributions. For a quantitative comparison of $I_t = 1$ and $I_t = 2$ we determine β at $t = m_\rho^2$ assuming that $\alpha = 1$ for both amplitudes. We obtain $\beta = 4.7 \pm 0.4$ for $I_t = 1$ and $\beta = 0.2 \pm 1.0$ for $I_t = 2$. In the superconvergent case β is compatible with zero, but the error is much larger. This is a general feature of FESR's: In amplitudes which are large (respectively, small) at high energies, the resonances enter with the same (respectively, alternating) signs, the amplitudes are therefore smooth (respectively, violently oscillating) at low energies, and the error in the FESR is small (respectively, large).

The $I_t = 0$ amplitude is complicated because it contains two Regge poles, called P and P', at $t = 0$. We use ρ and f as input on the LHS [case (II)], and take the cutoff $s_N = 1.92$ as above. At $t = m_f^2$ we obtain for the LHS $S_1 = 38$, while f contribution[9] to the RHS is 40 ± 9. Therefore one pole dominates ImA at $t = m_f^2$. (We do not know whether to identify the f with P or with P'.) In contrast P and P' have comparable importance at $t = 0$ (for $s = 1.9$ BeV^2), since the LHS = 1.8, while the P contribution[10] to the RHS gives 0.8, the difference evidently being due to the P'.

Finally we solve a simple bootstrap model. We use two equations: S_0 for $I_t = 1$ at $t = m_\rho^2$, and S_1 for $I_t = 0$ at $t = m_f^2$. We assume that the FESR are dominated by ρ and f. For algebraic convenience, (i) we put $m_\pi = 0$; (ii) we fix the cutoff N at the f resonance, and correspondingly take only half the f contribution on the LHS; (iii) we retain only the leading term in the Legendre function on both the LHS and RHS. We have two equations in the two unknowns: The mass ratio $\mu = (m_f/m_\rho)^2$ and the coupling ratio $\lambda = [(2l+1)xm\Gamma]_f/[\cdots]_\rho$. The ρ coupling merely fixes the scale of all amplitudes, while the ρ mass fixes the scale of all energies.[11] Finally we take $d\alpha/dt$ [which is needed in Eq. (4)] from experiment: $m_\rho^2(d\alpha/dt) = 0.60$. The equations S_0 and S_1 now read

$$\frac{1}{2}[3] + \frac{1}{3}\frac{\lambda}{2}\left[\frac{3}{2}\left(1+\frac{2}{\mu}\right)^2\right] = \frac{1}{2}\left(\mu + \frac{1}{2}\right)^2 \frac{0.60}{2} \times 2, \qquad (6)$$

$$1[1+2\mu]\left(1+\frac{\mu}{2}\right) + \frac{1}{3}\frac{\lambda}{2}\left[\frac{3}{2}(3)^2\right]\left(\frac{3\mu}{2}\right) = \frac{1}{4}\left(\frac{3\mu}{2}\right)^4 \lambda \frac{0.60}{2}\frac{3}{2}\left(\frac{2}{\mu}\right)^2. \qquad (7)$$

The solution is $\mu = 2.7$ and $\lambda = 2.0$, while the experimental values are $\mu = 2.7 \pm 0.2$ and $\lambda = 2.3 \pm 0.5$. If we restrict our attention to physical values $\lambda > 0$, $\mu > 0$ then the solution is unique and stable. Perturbing the LHS of (6) and (7) by 10% changes the solution by less than 10%.

Discussion of approximations and errors.— Resonance saturation on the LHS requires a low N, since above the low-energy region the leading direct-channel trajectories, ρ and f, will be accompanied by more and more resonances, or nonresonating background, in the lower partial waves. On the other hand, the assumption of Regge dominance on the RHS requires a high N.[12] As we go from $t = 0$ to $t = m_\rho^2$ the relative importance of the low partial waves decreases, since the contribution of each partial wave is proportional to $(2l+1) \times P_l(z_s)$. For example, for $s = m_\rho^2$ and $t = m_\rho^2$ we have $z_s \sim 3$. Therefore if the s-channel $\epsilon(750)$ exists and has the same width as the ρ, it will be only $\frac{1}{9}$ as important as the s-channel ρ at $t = m_\rho^2$. This relative suppression factor together with low widths or elasticities (or both) is responsible for the unimportance of the neglected resonances.[7] On the other hand, high partial waves become relatively more important as z_s increases. Because of this the real part of the partial-wave series diverges at $t = m_\rho^2$, so one might fear that the convergence[13] of the imaginary part is slow. Let us check how strong the first neglected wave is, for example the d wave at $s = m_\rho^2$. At $t = m_\rho^2$ and $s = m_\rho^2$, the Born d wave from ρ exchange (note that the Born approximation is good for high l) amounts to only 0.4% of the resonating ρ wave, and the d wave from the f tail amounts to 2%. Closely related is Bareyre's conclusion[14] for πN scattering that up to the 1688 resonance (F wave) all G-wave phase shifts are smaller than 3 deg. Evidently the prominent resonances are peripheral effects, $l_{\text{res}} \approx kR$, and the peripheral waves are either resonating or very small. We conclude that for $s = m_\rho^2$ and $t = m_\rho^2$ the ultraperipheral as well as the central partial waves are unimportant, and the LHS of the FESR is well approximated by the prominent peripheral resonance, the ρ. This is not surprising. The crucial point is that Regge theory in the direct channel tells us that for $t \to \infty$ the saturation of the LHS of the FESR by the leading resonances becomes exact.[15]

On the RHS we ask: For what N and t is a secondary Regge trajectory, ρ', negligible? We assume that the ρ' corresponds to particles in the t channel. These belong to low partial waves, and can be suppressed by going to high

z_t. This is equivalent to large N and/or low t. To summarize, on the LHS we want low N and/or high t, on the RHS we want high N and/or low t. Our quantitative analysis indicates that there is no gap between the two (s,t) regions in which the approximations are valid, e.g., at $s \approx t \approx m_\rho^2$ both approximations are good to 90%.

Unitarity is not used directly in the FESR bootstrap scheme, but it is eminently important in our estimates of the neglected terms, the ρ' in the t channel and the low-l background in the s channel.

The background integral in the l plane which was neglected on the RHS is responsible for the wiggles of the LHS as a function of N. We estimate the error from neglecting it by computing the standard deviation of the oscillating expression (LHS)$(\alpha+n+1)(N^{\alpha+n+1})^{-1}$ from its average value β. Numerical evaluation for $I_t=1$ in the region between the f and the g shows that this error amounts to about 10%, and that it can be simulated by taking $\delta N_1 = \pm 0.10$ BeV^2 in the narrow-resonance expression. In the narrow-resonance approximation the LHS becomes a step function, and the choice of N relative to adjoining resonances becomes importnat. For $I_t=1$ the narrow-resonance approximation reproduces the finite-width result, if we choose N halfway between adjoining resonances with $\delta N_2 = \pm 0.10$ BeV^2. Here we combine these two δN's and use $\delta N = 0.15$ BeV^2.

In the FESR bootstrap it is exact to work with only one channel, $\pi\pi \to \pi\pi$, and to leave out coupled channels like $\pi\pi \to K\bar{K}$, $\pi\pi \to \pi\omega$, etc. However the latter can give us additional information.

We should like to thank Professor G. Chew, Dr. G. Ringland, and Dr. J. Yellin for careful readings of the manuscript.

---

*This work was done under the auspices of the U. S. Atomic Energy Commission.

[1]D. Horn and C. Schmid, California Institute of Technology Report No. CALT-68-127, included in R. Dolen, D. Horn, and C. Schmid, Phys. Rev. 166, 1768 (1968).

[2]A. Logunov, L. D. Soloviev, and A. N. Tavkhelidze, Phys. Letters 24B, 181 (1967); K. Igi and S. Matsuda, Phys. Rev. Letters 18, 625 (1967).

[3]This bootstrap method was first proposed and applied by R. Dolen, D. Horn, and C. Schmid, Phys. Rev. Letters 19, 402 (1967). Related models: S. Mandelstam, Phys. Rev. 166, 1539 (1968); M. Ademollo, H. R. Rubinstein, G. Veneziano, and M. A. Virasoro, Phys. Rev. Letters 19, 1402 (1967); D. J. Gross, Phys. Rev. Letters 19, 1303 (1967).

[4]Within the framework of superconvergence relations there is no distinction between elementary and composite particles; see F. E. Low, in Proceedings of the Thirteenth International Conference on High-Energy Physics, Berkeley, 1966 (University of California Press, Berkeley, California, 1967).

[5]F. Zachariasen, Phys. Rev. Letters 7, 112, 268 (1961).

[6]Mandelstam, Ref. 3.

[7]A. H. Rosenfeld et al., Rev. Mod. Phys. 39, 1 (1967). We assume that the g has spin 3, because its mass coincides with the expected recurrence of the ρ.

[8]V. Barger and L. Durand, III, Phys. Rev. Letters 19, 1295 (1967); L. Serterio and M. Toller, Phys. Rev. Letters 19, 1146 (1967).

[9]Note that the $f'(1500)$ is unimportant compared with the $f(1250)$ because $x_{f'} < 0.14$, $\Gamma_{f'} < \Gamma_f$, and $\alpha_{f'}(t) < \alpha_f(t)$.

[10]W. Rarita, R. J. Riddell, C. B. Chiu, and R. J. N. Phillips, Phys. Rev. 165, 1615 (1968).

[11]To fix the mass scale we can use either the ρ mass or the (universal?) slope of Regge trajectories α'. The π mass is too small to be useful. Putting it equal to zero introduces errors not larger than 10% in the various FESR's.

[12]V. Barger and R. J. N. Phillips, Phys. Letters 25B, 351 (1967), were so impressed by the requirement of large N for the RHS that they chose $p_N = 4.15$ BeV/c. At so high an N value, the resonance approximation to the integrand on the LHS is too small by more than a factor of 10, since at these energies the leading resonances have "exponentially" decreasing elasticities.

[13]We assume that the ρ pole at $t = m_\rho^2$ is much more important than the double spectral function. We therefore neglect the latter.

[14]P. Bareyre, C. Bricman, and G. Villet, to be published.

[15]For $t \to +\infty$ the double spectral function ρ_{st} is therefore correctly included, although we ignore the threshold singularity of ρ_{st} at $t = 4m_\pi^2$.