[2]A. C. Anderson, D. O. Edwards, W. R. Roach, R. E. Sarwinski, and J. C. Wheatley, Phys. Rev. Letters 17, 367 (1966).

[3]P. M. Platzman and P. A. Wolff, Phys. Rev. Letters 18, 280 (1967).

[4]L. D. Landau, Zh. Eksperim. i Teor. Fiz. 30, 1058 (1956) [translation: Soviet Phys.–JETP 3, 920 (1957)].

[5]V. P. Silin, Zh. Eksperim. i Teor. Fiz. 33, 1227 (1957) [translation: Soviet Phys.–JETP 6, 945 (1958)].

[6]In what follows we put the spin relaxation times T_1 and T_2 equal to infinity.

[7]D. Hone, Phys. Rev. 121, 669 (1961). The difference between the diffusion coefficients governing the decay of J_z and J^+ entails complications which there is no space to discuss here; Eq. (11) is simply to be regarded as a convenient definition of τ_D.

[8]This is not quite obvious if $Z_1 \neq 0$ and deserves theoretical and experimental checking. Our preliminary calculations appear to indicate that it is so.

[9]For details of the theory we refer to the standard treatments; provided the detector is not phase-sensitive the new features introduced here give rise to no special difficulty.

[10]Cf. A. J. Leggett, to be published.

[11]J. Bardeen, G. Baym, and D. Pines, Phys. Rev. 156, 1, 207 (1967).

[12]Or in principle without any approximation from the density of the normal component at $T=0$.

DEPENDENCE OF CRITICAL PROPERTIES ON DIMENSIONALITY OF SPINS

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We consider a new model Hamiltonian $\mathcal{H}^{(\nu)}$ for interacting ν-dimensional classical "spins"; $\mathcal{H}^{(\nu)}$ reduces to the Ising, planar, and Heisenberg models, respectively, for $\nu=1$, 2, and 3. Certain critical properties of $\mathcal{H}^{(\nu)}$ are found to be monotonic functions of ν.

Although it was first introduced as a simple model of ferromagnetism, the nearest-neighbor $S=\frac{1}{2}$ Ising model has come to serve as a practical model for a binary alloy and a classical gas.[1] More recently, the classical planar model has received attention as a fairly crude lattice model for the λ transition in a Bose fluid.[2,3] The classical Heisenberg model has been proposed[4] as a realistic model for isotropically interacting spins at temperatures in the neighborhood of the critical temperature T_c. Finally, the spherical model[5] has long been attractive, especially since it is exactly soluble. The Ising, planar, and Heisenberg models are special cases (for $\nu=1$, 2, and 3) of

$$\mathcal{H}^{(\nu)} = -2J \sum_{\langle ij\rangle} \vec{S}_i^{(\nu)} \cdot \vec{S}_j^{(\nu)}, \tag{1}$$

where the $\vec{S}_i^{(\nu)}$ are ν-dimensional vectors of magnitude $\sqrt{\nu}$, and $-2J\nu$ is the energy of a nearest-neighbor pair $\langle ij\rangle$ of parallel "spins" localized on sites i and j. Here we apply high-temperature expansion methods to obtain some critical properties of the model Hamitonian (1).

We have calculated, for general ν and for general lattice structure, the coefficients $\hat{a}_n^{(\nu)}$ in the expansion for the zero-field reduced susceptibility

$$\bar{\chi}^{(\nu)} = 1 + \sum_{n=1}^{\infty} \hat{a}_n^{(\nu)} x^n \tag{2}$$

[through order $n=8$ for close-packed and through order $n=9$ for loose-packed lattices], and the coefficients $\hat{c}_n^{(\nu)}$ in the specific-heat series

$$C^{(\nu)} = Nk \sum_{n=2}^{\infty} \hat{c}_n^{(\nu)} x^n \tag{3}$$

[through orders $n=9$, 10 for close- and loose-packed lattices, respectively]. Here $x\equiv 2J/kT$ and k is the Boltzmann constant. The coefficients were computed directly from a diagrammatic representation of the zero-field spin correlation function $\langle \vec{S}_0 \cdot \vec{S}_R\rangle_\beta^{(\nu)}$ for $\mathcal{H}^{(\nu)}$.[6] The requisite diagrams[7] are the same regardless of the dimensionality of the spins. Moreover, certain topological similarities among the diagrams may be exploited to reduce from 298 to only 15 the number of averages or "traces" which must actually be calculated! We obtain complete agreement with previous calculations[1,3,4] of the coefficients $\hat{a}_n^{(\nu)}$ and $\hat{c}_n^{(\nu)}$ when we specialize to the cases $\nu=1$, 2, and 3; hence our work serves as an independent and very thorough check on these previous calculations. Moreover, because only one diagrammatic calculation is required to obtain the expansions for general ν, it is quite feasible to extend to arbitrary ν several calculations which have recently been performed for $\nu=1$ (the $S=\frac{1}{2}$ Ising model).[8]

It has proved possible to solve exactly for the partition function and spin-correlation function of a linear chain with N ν-dimensional spins, from which we obtain the exact expressions $\bar{\chi}^{(\nu)}=(1+y_\nu)/(1-y_\nu)$ for the reduced susceptibility and $E^{(\nu)}=-2J\nu y_\nu$ for the internal energy per spin (in the limit $N\to\infty$). Here

$$y_\nu \equiv \frac{1}{\nu}\frac{\partial}{\partial x}\ln[x^{\frac{1}{2}\nu-1}I_{\frac{1}{2}\nu-1}(\nu x)];$$

thus, e.g., $y_1=\tanh x$, $y_2=I_1(2x)/I_0(2x)$, $y_3=\mathcal{L}(3x)\equiv\coth 3x-1/3x$, and $y_\infty=2x/[1+(1+4x^2)^{1/2}]$.[9]

These exact results for the [1]-dimensional lattice have motivated us to calculate general-lattice expressions for the coefficients in the (generally smoother) series

$$\bar{\chi}^{(\nu)}=1+\sum_{n=1}^{\infty}A_n^{(\nu)}y_\nu^{\,n} \tag{4}$$

and

$$E^{(\nu)}=-(J/y_\nu)\sum_{n=2}^{\infty}B_n^{(\nu)}y_\nu^{\,n}. \tag{5}$$

No new diagrammatic calculation is necessary to get the new coefficients in Eqs. (4) and (5),[10] e.g., the $A_n^{(\nu)}$ are obtained directly from the $\hat{a}_n^{(\nu)}$ of Eq. (2) by substituting in Eq. (4) the small-x (high-temperature) expansion of y_ν,

$$y_\nu=x[1+Q_1^{(\nu)}x^2+Q_2^{(\nu)}x^4+Q_3^{(\nu)}x^6+\cdots], \tag{6}$$

and then equating coefficients of successive powers of x in Eqs. (2) and (4).[11] Here $Q_1^{(\nu)}=-\nu/(\nu+2)$, $Q_2^{(\nu)}=2\nu^2/[(\nu+4)(\nu+2)]$, $Q_3^{(\nu)}=-\nu^3(5\nu+12)/[(\nu+6)(\nu+4)(\nu+2)^2]$, $Q_4^{(\nu)}=2\nu^4(7\nu+24)/[(\nu+8)(\nu+6)(\nu+4)(\nu+2)^2]$, etc.

Having calculated the general-ν expressions for the basic coefficients in Eqs. (2)-(5), we proceed to study the dependence of various critical properties upon the dimensionality of the spins. First consider the critical temperatures $T_c^{(\nu)}$ for [d]-dimensional lattices. For the [1]-dimensional linear chain we find that, when applied to the expansion (2), neither Padé approximant methods[12] nor "ratio" methods[13] predict the known result $T_c^{(\nu)}=0$.[14] However the new expansion coefficients $A_n^{(\nu)}=z\sigma^{n-1}$ for all ν, and we correctly extrapolate to $y_\nu=1$ (or $T_c^{(\nu)}=0$). For [2]- and [3]-dimensional lattices, ratio and Padé methods applied to either Eq. (2) or (4) suggest that $T_c^{(\nu)}$ decreases smoothly and monotonically from its value at $\nu=1$ to its value at $\nu=\infty$ [see Fig. 1]. The critical exponent $\gamma^{(\nu)}$ in the assumed form of the divergence of the susceptibility, $\bar{\chi}\sim(T-T_c)^\gamma$, correspondingly increases as ν goes from 1 to ∞. It appears that $\gamma^{(\nu)}$ is independent of lattice within the class of fcc, bcc, and sc lattices,[15] and that its variation with ν [see Fig. 2] from its $\nu=1$ value (~1.25) to its exact $\nu=\infty$ value (2.0) may be conveniently summarized to within a few percent by the mnemonic formula $\gamma^{(\nu)}=1+\tanh[(\nu+3)/16]$. The specific-heat exponent $\alpha^{(\nu)}$ also appears to vary smoothly from its $\nu=1$ value ($\frac{1}{8}$) to its $\nu=\infty$ value (-1).[16] Similarly, as ν goes from 1 to ∞, the exponent $\tilde{\nu}$ [describing the approach to zero as $T\to T_c^+$ of the inverse correlation range, $\kappa\sim(T-T_c)^{\tilde{\nu}}$] increases smoothly from 0.64 to 1.

In summary, we have introduced a new model Hamiltonian for both computational and conceptual reasons. Computationally, we found that we could obtain high-temperature expansions based upon (1) for arbitrary ν with no more difficulty than for, say, $\nu=3$; moreover, the exceedingly difficult (in practice) problems

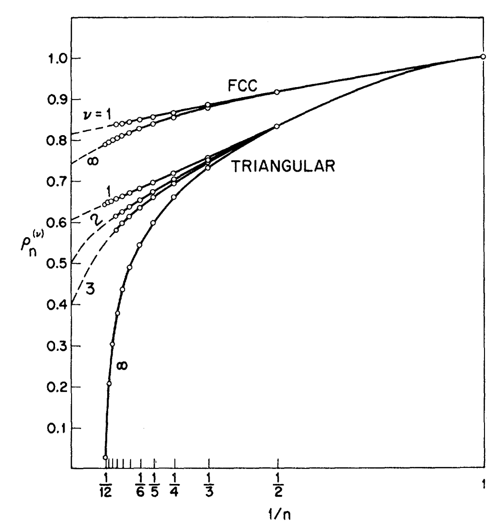

FIG. 1. Variation of $T_c^{(\nu)}$ with ν for the fcc and plane triangular lattices. The ratios $\rho_n^{(\nu)}\equiv\hat{a}_n^{(\nu)}/\hat{a}_1^{(\nu)}\hat{a}_{n-1}^{(\nu)}$ of successive coefficients $\hat{a}_n^{(\nu)}$ in the susceptibility expansion (2) are plotted against $1/n$. As $n\to\infty$, $\rho_n^{(\nu)}\to t_c^{(\nu)}\equiv kT_c^{(\nu)}/2zJ$. Three of the extrapolations shown (dashed lines) are to exact values: $t_c^{(\infty)}$[fcc] $=0.7436\cdots$, $t_c^{(1)}[\Delta]=0.6068\cdots$, and $t_c^{(\infty)}[\Delta]=0$. Curves for other values of ν lie between the ones shown.

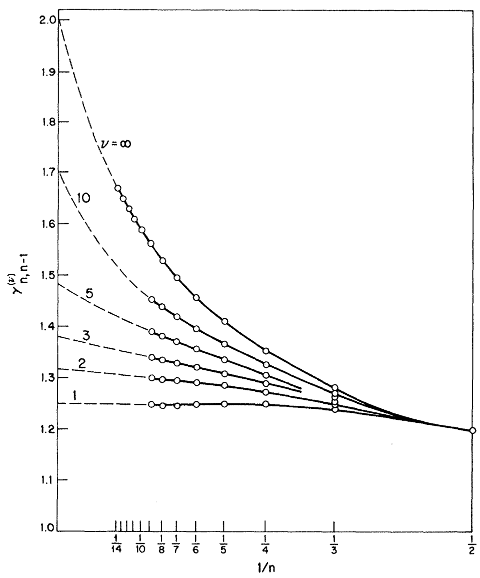

FIG. 2. Variation of $\gamma^{(\nu)}$ with ν for the fcc lattice. The function $\gamma_{n,n-1}^{(\nu)} \equiv 1-n+n\rho_n^{(\nu)}/t_{n,n-1}^{(\nu)}$, where $t_{n,n-1}^{(\nu)} \equiv n\rho_n^{(\nu)}-(n-1)\rho_{n-1}^{(\nu)}$, gives the value of γ which would be obtained by placing a straight line through the ratios $\rho_n^{(\nu)}$ and $\rho_{n-1}^{(\nu)}$ (plotted against $1/n$ as, e.g., in Fig. 1). Thus $\gamma_{n,n-1}^{(\nu)}$ should approach $\gamma^{(\nu)}$ as $n\to\infty$ if χ diverges with a power law as $T\to T_c^{(\nu)}$ One of the extrapolations shown is exact: $\gamma^{(\infty)}=2$. Curves for other values of ν lie between the ones shown.

of checking a calculation are substantially alleviated, since the limit $\nu=\infty$ is exactly soluble[5] (as is the case $\nu=1$ for [2]-dimensional lattices). Conceptually, we have found that certain critical properties do appear to depend upon ν (contrary to the belief that "detailed properties of the Hamiltonian become unimportant in the critical region"), but that this ν dependence is smooth and monotonic. It is of course quite possible that for two-dimensional lattices $T_c^{(\nu)}$ is nonzero only for $\nu=1$ (the $S=\frac{1}{2}$ Ising model).[17]

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[1]Over 400 articles have been published on the Ising model. A very recent survey, together with references to many of the important papers as well as to most review articles, is S. G. Brush, Rev. Mod. Phys. 39, 883 (1967).

[2]V. G. Vaks and A. I. Larkin, Zh. Eksperim. i Teor. Fiz. 49, 975 (1965) [translation: Soviet Phys.—JETP 22, 678 (1966)]. The isotropic classical planar model is not a rigorous model for the λ transition, but is of interest as a well-defined statistical mechanical problem.

[3]R. G. Bowers and G. S. Joyce, Phys. Rev. Letters 19, 630 (1967).

[4]H. E. Stanley and T. A. Kaplan, Phys. Rev. Letters 16, 981 (1966); G. S. Joyce and R. G. Bowers, Proc. Phys. Soc. (London) 88, 1053 (1966); P. J. Wood and G. S. Rushbrooke, Phys. Rev. Letters 17, 307 (1966).

[5]As $\nu\to\infty$, it has been seen that $\mathcal{H}^{(\nu)}$ is essentially equivalent to the spherical model (H. E. Stanley, to be published). All that we refer to in this Letter is the equivalence for $T>T_c$; the situation for $T<T_c$ requires further comment and will be discussed elsewhere.

[6]The procedure of obtaining the $\hat{a}_n^{(\nu)}$ from the spin correlation function is illustrated for the case $\nu=3$ in H. E. Stanley, Phys. Rev. 158, 546 (1967).

[7]The 298 diagrams [which are required to carry the expansions (2) and (3) as far as we have] are explicitly displayed in Figs. 1-5 of H. E. Stanley, Phys. Rev. 158, 537 (1967). These diagrams were used to calculate the "second moment series" $\sum_R R^2 \langle \vec{S}_0 \cdot \vec{S}_R \rangle_\beta$ (required to determine the exponent $\tilde{\nu}$ describing the inverse correlation range) as well as the susceptibility $(\sum_R \langle \vec{S}_0 \cdot \vec{S}_R \rangle_\beta)$ series (2) and the specific-heat (~nearest-neighbor correlation function) series (3).

[8]For example, M. F. Sykes, J. L. Martin, and D. L. Hunter [Proc. Phys. Soc. (London) 91, 671 (1967)] have very recently extended the series (3) for the $S=\frac{1}{2}$ Ising model; similarly, N. W. Dalton [Proc. Phys. Soc. (London) 88, 659 (1966)] has included next-nearest neighbors in the series (2) for the case $\nu=1$. We have already extended to arbitrary ν the calculation of the second-moment series (see Ref. 7).

[9]Here the $I_\nu(x)$ are modified Bessel functions of the first kind.

[10]Another expansion which can be obtained without any new diagrammatic calculation is motivated by the Bethe-Peierls approximation, $\overline{\chi}^{(\nu)}=(1+y_\nu)/(1-\sigma y_\nu)$, where $\sigma\equiv z-1$, and z is the number of nearest neighbors. Suppose that we "factor out" the Bethe-Peierls singularity from our expansion (4):

$$\overline{\chi}^{(\nu)}=(1-\sigma y_\nu)^{-2}[1-(\sigma-1)y_\nu-\sigma y_\nu^2+\sum_{n=3}^{\infty} D_n^{(\nu)} y_\nu^n].$$

The new coefficients $D_n^{(\nu)}$ are thus defined in terms of the $A_n^{(\nu)}$ of Eq. (4) by the recursion relation $D_n^{(\nu)}=A_n^{(\nu)}-2\sigma A_{n-1}^{(\nu)}+\sigma^2 A_{n-2}^{(\nu)}$. Whereas the coefficients $A_n^{(\nu)}$ are very complicated functions of the various lattice constants, the $D_n^{(\nu)}$ are quite simple.

[11]Thus, e.g., $A_1^{(\nu)}=\hat{a}_1^{(\nu)}=z$; $A_2^{(\nu)}=\hat{a}_2^{(\nu)}=z\sigma$; $A_3^{(\nu)}=\hat{a}_3^{(\nu)}-Q_1^{(\nu)}A_1^{(\nu)}=z\sigma^2-6p_3$; $A_4^{(\nu)}=\hat{a}_4^{(\nu)}-2Q_1^{(\nu)}A_2^{(\nu)}$

$= z\sigma^3 - 8p_4 - 12p_3[\sigma + (\nu-1)/(\nu+2)]$.

[12]See, e.g., the comprehensive review article, G. A. Baker, Jr., Advan. Theoret. Phys. 1, 1 (1965), and references therein.

[13]C. Domb and M. F. Sykes, Phys. Rev. 128, 168 (1962).

[14]There is one exception: For $\nu = 1$, the ratios $\rho_n \equiv a_n/a_1 a_{n-1} = 1/n$, so that extrapolation to $T_c^{(1)} = 0$ is possible even with the [generally less regular] series (2).

[15]We also note that, in general, $\gamma^{(\nu)}$ is not a simple fraction—though of course certain values may well be (e.g., $\gamma^{(2)} \cong 1.32 \cong 21/16$, $\gamma^{(3)} \cong 1.38 \cong 11/8$).

[16]Actually the $T \to T_c^+$ singularity in the specific heat need not occur as a simple factor, e.g., for the fcc lattice, $C^{(1)} \sim 1.09(1-T_c/T)^{-1/8} - 1.24$ [M. F. Sykes et al., Ref. 8]; $C^{(\infty)} \sim \text{const}(1-T_c/T) + k/2$. We note that $\alpha^{(\nu)}$ appears to pass through zero (corresponding to a logarithmic singularity) between $\nu = 2$ and $\nu = 3$; for $\nu > 3$, α is negative and the specific heat has a cusplike singularity. In particular, $\alpha^{(3)}$ appears to be slightly less than zero (-0.03 ± 0.03), which is consistent with the most recent measurements on the "Heisenberg ferromagnet" EuS, for which α was found to be 0 ± 0.03 (B. J. C. van der Hoeven, D. T. Teaney, and V. L. Moruzzi, to be published). Note that if we define the magnetization exponent $\beta^{(\nu)}$ by the (nonrigorous) equality $\alpha^{(\nu)} + 2\beta^{(\nu)} + \gamma^{(\nu)} \equiv 2$ [not to be confused with the rigorous Rushbrooke inequality $\alpha' + 2\beta + \gamma' \geq 2$], we find that $\beta^{(\nu)}$ also varies smoothly and monotonically from its accepted $\nu = 1$ value to its exact $\nu = \infty$ value: $\beta^{(1)} \cong 5/16$, $\beta^{(2)} \cong 11/32$, $\beta^{(3)} \cong \frac{1}{3}, \cdots, \beta^{(\infty)} = \frac{1}{2}$. [The critical exponents are defined and discussed in Ref. 1.]

[17]Indeed, the indicated phase transitions, if real (for there is nothing rigorous about extrapolating from a finite number of terms of an infinite series), would have to be a new type of low-temperature phase with no infinite-range order M, yet with sufficient long-range order that χ diverges to infinity.