CURRENT ALGEBRA: A SIMPLE MODEL WITH NONTRIVIAL MASS SPECTRUM

H. Leutwyler
Institut für theoretische Physik der Universität Bern, Bern, Switzerland
(Received 10 October 1967)

In the present Letter we exhibit a relativistically invariant solution of the local isovector current algebra at infinite momentum. This solution is based on the assumption that in the infinite-momentum limit the algebra is saturated by one-particle states.[1] Actually, it turns out that this assumption is satisfied only in a formal manner: The space of state vectors contains—in addition to a discrete set of physical one-particle states—a continuum of ghost states with spacelike momentum. This situation is by no means characteristic of the particular model to be described below; it arises for a large class of solutions of the algebra,[2] and it is doubtful whether solutions with nontrivial mass spectrum exist that do not contain a continuum of such ghost states. We shall comment on the significance of these ghosts at the end of the paper.

In the following we restrict ourselves to the case of a set of particles of one and the same isospin but allow the spin of these particles to be arbitrary. This restriction implies that the dependence of the matrix elements on the isospin quantum numbers is trivial and can be factored out. In the infinite-momentum limit the reduced matrix elements then satisfy[1,3]

$$v(\vec{q})v(\vec{q}\,')=v(\vec{q}+\vec{q}\,'). \tag{1}$$

Relativity implies two very stringent restrictions on the solutions of this equation. We quote the infinitesimal version of these restrictions[1,3]:

$$[J_3,v(\vec{q})]=i\vec{q}\times\nabla_q v(\vec{q}), \tag{2}$$

$$\{I,\{I,\{I,v(\vec{q})\}\}\}=\tfrac{1}{4}[M^2,[M^2,\{I,v(\vec{q})\}]]+\tfrac{1}{2}\vec{q}^2[M^2,\{I,v(\vec{q})\}]_+ +\tfrac{1}{4}(\vec{q}^2)^2\{I,v(\vec{q})\}. \tag{3}$$

Here the symbol $\{I,X\}$ stands for

$$\{I,X\}=\tfrac{1}{2}[M^2,[J_3,X]]-\tfrac{1}{2}\vec{q}^2[J_3,X]_+-[\vec{q}\cdot M\vec{J},X].$$

The bracket $[\,,\,]_+$ indicates an anticommutator, M is the mass operator, and J_1, J_2, J_3 generate the rotation group. We wish to construct operators $v(\vec{q})$, M^2, $M\vec{J}$, and J_3 that satisfy this algebra and in addition obey the conventional commutation rules of the rotation group. In the following, we briefly sketch the heuristic procedure which led us to the solution to be described below.

Clearly, Eq. (1) is satisfied if $v(\vec{q})$ is of the form

$$v(\vec{q})=\exp i\vec{q}\cdot\vec{x}, \tag{4}$$

where x_1 and x_2 are two commuting Hermitian operators. Equation (2) states that $\vec{x}$ transforms like a vector under J_3. To extract some information from Eq. (3) we extend the real vector $\vec{q}$ to complex values and consider the particular case $\vec{q}=p(1,i)$. This particular vector satisfies $\vec{q}^2=0$, and hence Eq. (3) simplifies considerably. It can be shown that the resulting equation is satisfied to all orders in p provided the following simple relations are fulfilled[4]:

$$[[[M^2,x_+],x_+],x_+]=0, \tag{5}$$

$$[M^2,x_+^2]=-4i[MJ_+,x_+]. \tag{6}$$

Here x_+ and J_+ stand for x_1+ix_2 and J_1+iJ_2, respectively. To solve these equations we assume that the representation of the two commuting operators x_1 and x_2 is irreducible[5] in the sense that an operator commuting with both x_1 and x_2 is a function of x_1 and x_2. In this case the simultaneous eigenstates $|\vec{x}\rangle$ of these operators form a complete set of basis vectors. Equation (5) then suggests that in this representation M^2 is a local differential operator of second order and by virtue of (6) the same must be true for MJ_+. We thus look for a solution of (5) and (6) in terms of second-order differential operators which are furthermore required to satisfy

$$[MJ_+,M^2]=0,$$
$$[MJ_+,MJ_-]=2M^2J_3. \tag{7}$$

It is straightforward, but quite laborious, to check that the following expressions satisfy these commutation rules:

$$M^2=-\tfrac{1}{2}[\alpha,\Delta]_+ +\beta,$$

$$M\vec{J}=\tfrac{1}{2}i[J_3,\vec{\partial}]_+ +\tfrac{1}{4}[\Delta,(\alpha+1)\vec{x}_*]_+ -\tfrac{1}{2}\gamma\vec{x}_*,$$

$$J_3=-i\vec{x}\times\vec{\partial}. \tag{8}$$

Here $\vec{\partial}$ stands for the operator $(\partial/\partial x_1, \partial/\partial x_2)$, Δ denotes the Laplacian $\partial^2/\partial x_1^2 + \partial^2/\partial x_2^2$, and the vector $\vec{x}_*$ is defined by $\vec{x}_* = (x_2, -x_1)$. The functions α, β, and γ are given by

$$\alpha = \mu r(1-\mu r)^{-1},$$
$$\beta = \tfrac{1}{4}\alpha r^{-2}(1+\alpha)^2 + \kappa(1+\alpha),$$
$$\gamma = \beta + \tfrac{1}{4}r^{-2}(1+\alpha)^2, \qquad (9)$$

where r denotes the length of the vector $\vec{x}$, $r = (x_1^2 + x_2^2)^{1/2}$. These expressions involve the two parameters μ and κ which represent the only degrees of freedom in our solution. It can be shown that the operators given above, in fact, satisfy the full invariance condition (3) and thus constitute a relativistically invariant current algebra.

To interpret our solution physically we consider simultaneous eigenstates of the mass operator and of J_3:

$$M^2\psi(\vec{x}) = m^2\psi(\vec{x}), \qquad (10)$$
$$J_3\psi(\vec{x}) = j_3\psi(\vec{x}). \qquad (11)$$

Expressed in polar coordinates by $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$, the eigenfunctions of J_3 are of the form

$$\psi(\vec{x}) = (r/\alpha)^{1/2}\Phi(r)\exp ij_3\varphi, \qquad (12)$$

where we have extracted the normalization factor $(r/\alpha)^{1/2}$ to simplify the resulting radial Eq. (10) which now reads

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{1}{r^2}(j_3^2 - \tfrac{1}{4}) + \frac{1}{\mu r}(m^2-\kappa) - m^2\right] \times \Phi(r) = 0. \qquad (13)$$

It is amusing to note that this equation is in fact identical with the radial equation for the hydrogen atom with the peculiarity that the strength of the Coulomb potential depends on the eigenvalue m^2. In terms of the conventional quantum number n, the discrete mass eigenvalues are

$$m = \mu n \pm (\kappa + \mu^2 n^2)^{1/2}. \qquad (14)$$

Furthermore, j_3 is related to the usual symbol l by

$$|j_3| = l + \tfrac{1}{2}. \qquad (15)$$

The fact that the hydrogen atom possesses an accidental degeneracy is of crucial importance in the present context,[6] because the given eigenvalue m^2 must correspond to $2j+1$ eigenstates with different eigenvalues of J_3. Indeed there are two states corresponding to $n=1$: $j_3 = \pm\frac{1}{2}$. These two states clearly constitute a $j=\frac{1}{2}$ multiplet. The next level, $n=2$, contains $l = 0, 1$. According to (15) there are four states belonging to these values of l: $j_3 = \pm\frac{1}{2}, \pm\frac{3}{2}$; these states constitute a $j = \frac{3}{2}$ multiplet and so on. In general,

$$j = n - \tfrac{1}{2}. \qquad (16)$$

We thus find that the representation contains an infinite sequence of particles with spin $j = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots$. This sequence of spin values already occurs for the degenerate solution $M^2 = \text{const}$, which is associated with a unitary representation of the group $SL(2, C)$.[7] Indeed, we recover the degenerate solution in the limit $\mu \to 0$. In the general case the mass spectrum of the above sequence of states is given by (14) with $n = j + \frac{1}{2}$. This spectrum is quite reasonable provided $\mu > 0$ and $\kappa > -\mu^2$ and provided we choose the plus sign in front of the square root.[8]

In addition to the bound states just discussed, the model also contains a continuum of scattering states[9] associated with negative values of m^2, i.e., spacelike momentum. The current algebra is not satisfied if one restricts oneself to the subspace spanned by the bound states, since the current produces transitions between bound and scattering states.

To clarify the role played by the ghost states with spacelike momentum, it is instructive to study the behavior of the solution as the mass-splitting parameter μ tends to zero. In this limit, the bound states go over smoothly into the corresponding states of the degenerate solution, whereas the ghost states do not have a well-defined limit; the matrix elements of the current between bound and scattering states are not analytic at $\mu = 0$. It can be shown that the power series expansion to any finite order in μ of the current matrix elements between bound states satisfies the current-algebra relations to the same order in μ. In this sense of an asymptotic expansion to any finite order in the mass splitting the ghosts may therefore be ignored.

The occurrence of ghosts for finite values of the mass splitting presents a kind of phase transition which is possible only because the model contains infinitely many states. We conclude that the ghosts are due to the failure of the one-particle approximation in the high-energy region, where the spectrum presumably consists of broad resonances with many decay channels, poorly represented by an infinite set of sharp lines.

The inclusion of many-particle states seems unavoidable if ghosts are to be eliminated in an exact solution with finite mass splitting. Nevertheless, the simple model presented here may provide a useful approximation to the extent that the power series expansion of the current matrix elements between bound states to some given order in the mass splitting reproduces the exact formal solution.

The author is indebted to Dr. H. Bebié, Dr. V. Gorgé, and F. Ghielmetti for useful remarks and stimulating discussions.

---

[1]For a derivation of the basic equations (1)-(3) we refer the reader to the literature of R. F. Dashen and M. Gell-Mann, Phys. Rev. Letters 17, 340 (1966), and in Proceedings of the Third Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, 1966, edited by A. Perlmutter, J. Wojtaszek, G. Sudarshan, and B. Kurşunoğlu (W. H. Freeman & Company, San Francisco, California, 1966); J. Weyers, Phys. Rev. Letters 18, 1033 (1967); H. Bebié and H. Leutwyler, Phys. Rev. Letters 19, 618 (1967).

[2]After the completion of this work we received a preprint by M. Gell-Mann, D. Horn, and J. Weyers, to be published, who describe similar models. Close examination of these models reveals that they also contain ghost states.

[3]The vector symbol is reserved for two-component objects. In particular, the vector $\vec{q}=(q_1, q_2)$ denotes the transverse momentum transfer.

[4]These commutation rules have been derived independently by M. Gell-Mann et al., in Proceedings of the Heidelberg Conference on High Energy Physics and Elementary Particles, Heidelberg, Germany, 1967 (to be published). There Gell-Mann et al. also conjectured a mass spectrum of the type (14).

[5]This assumption characterizes our model. In the models described in Ref. 2 the representation of the operators x_1 and x_2 is reducible.

[6]If one includes the dependence of the wave functions on the angular variable φ, our solution corresponds more properly to the two-dimensional hydrogen atom. The operator $M\vec{J}$ is an analog of the Runge-Lenz vector associated with the Coulomb problem in two dimensions. The rather involved structure of $M\vec{J}$ as compared with the Runge-Lenz vector arises because we not only imposed the requirement that $M\vec{J}$ commutes with the "Hamiltonian," but in addition that it satisfies the second commutation relation in (7).

[7]S. Fubini, in Proceedings of the Fourth Coral Gables Conference on Symmetry Principles at High Energies, University of Miami, January, 1967, edited by A. Perlmutter and B. Kurşunoğlu (W. H. Freeman & Company, San Francisco, California, 1967). See also Ref. 1.

[8]If $\mu > 0$, the representation of the algebra is only a formal one, since in this case the function α possesses a singularity at $r=\mu^{-1}$. A possible way to give a precise meaning to the solution for $\mu > 0$ is the following: One first considers negative values of μ, then computes the matrix elements of interest in the angular momentum representation $|j,j_3\rangle$, and at the end continues analytically in μ. I am indebted to F. Ghielmetti for this remark.

[9]I am indebted to Professor Fierz for a discussion on this point.