[5]The data presented here are only those from the first series of runs made: 0.32 and 4.2°K. Other data taken at different temperatures will be presented in a later publication.

[6]J. P. Scanlon, G. H. Stafford, J. J. Thresher, and P. H. Bowen, Rev. Sci. Instr. 28, 749 (1957).

[7]E. Ambler, R. B. Dove, and R. S. Kaeser, Advances in Cryogenic Engineering (Plenum Press, Inc., New York, 1963), p. 443; E. Ambler, E. G. Fuller, and H. Marshak, Phys. Rev. 138, B117 (1965).

[8]This will be described in more detail in a later publication.

[9]Since we are using a polycrystalline sample similar to Ref. 3, we will follow their convention and use the statistical tensor notation.

[10]R. S. Safrata, T. R. Fisher, and E. G. Shelley, J. Appl. Phys. 37, 4869 (1966).

[11]These measurements were kindly made for us by Dr. S. Foner of the National Magnet Laboratory.

[12]S. Kobayashi, H. Kamitsubo, K. Katori, A. Uchida, M. Imaizumi, and K. Nagamine, J. Phys. Soc. Japan 22, 368 (1967), also Progr. Theoret. Phys. (Kyoto) 36, 1071 (1966).

[13]D. G. Foster, Jr., and D. W. Glasgow, private communication (1967).

[14]P. H. Bowen, J. P. Scanlon, G. H. Stafford, J. J. Thresher, and P. E. Hodgson, Nucl. Phys. 22, 640 (1961).

[15]We are indebted to Dr. F. G. Perey for these studies.

[16]B. Elbek, Determination of Nuclear Transition Probabilities by Coulomb Excitation (Ejnar Munksgaard Forlag, Copenhagen, 1963); M. Danos and W. Greiner, Phys. Letters 8, 113 (1964); E. Ambler, E. G. Fuller, and H. Marshak, Phys. Rev. 138, B117 (1965).

[17]T. Tamura, Rev. Mod. Phys. 37, 679 (1965).

[18]T. Tamura, Oak Ridge National Laboratory Report No. ORNL-4152, 1967 (unpublished).

[19]We are indebted to Dr. A. Prince of the Brookhaven National Laboratory for performing these calculations with the CDC 6600 computer.

[20]Actually Perey's search run, Ref. 15, was made for $E_n = 10$ to 120 MeV, and we extrapolated his parameters to $E_n < 10$ MeV. This is the reason why the fit is poorer for these lower energies. An extension of the search run to $E_n < 10$ MeV is underway and it is hoped that a better fit will be obtained over the entire range of E_n.

[21]J. D. Lawson, Phil. Mag. 44, 102 (1953); J. M. Peterson, Phys. Rev. 125, 955 (1962).

UNITARITY CORRECTIONS TO LOW-ENERGY PARAMETERS IN SOFT-PION CALCULATIONS

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We unitarize the results of soft-pion current-algebra calculations of low-energy parameters for s-wave pion-nucleon scattering in the $I=\frac{1}{2}$ channel, using the N/D formalism. We isolate clearly corrections arising from unitarity, off-shell kinematics, and off-shell dynamics.

Considerable success has been achieved in recent years in understanding a wide range of scattering and decay phenomena by use of the chiral algebra of currents[1] in conjunction with the hypothesis of partial conservation of axial-vector currents (PCAC).[2] We recall in particular the calculations of pion-nucleon scattering lengths by Weinberg,[3] Tomozawa,[4] and others.[5]

Such calculations of low-energy parameters in elastic scattering have several characteristic features. Use of the PCAC hypothesis necessitates a consideration of amplitudes with both initial and final pions off their mass shells. In each partial wave and isospin channel one is thus led to consider functions of the form $f_{l\pm}{}^{I}(s,q^2)$, where s is the square of the total center-of-mass energy in the direct channel and q^2 is the variable mass of the off-shell pions (initial and final pions are assumed to have the same mass). For comparison with experiment, the amplitude is evaluated at the point $q^2=0$. Weinberg combined the off-shell limit with the soft-pion limit $q_\mu=0$. Schnitzer[6] subsequently investigated the amplitude in the same off-shell limit without making the pions soft. In both cases, smooth extrapolation to $q^2=m_\pi^2$ is assumed, which allows comparison of threshold parameters evaluated at $s=m_N^2$ against experimental values.

We adopt the viewpoint that the extrapolation of low-energy parameters in pion-nucleon scattering evaluated at the unphysical points $s=m_N^2$ and $q^2=0$ to the physical points $s=(m_N+m_\pi)^2$

and $q^2 = m_\pi^2$ requires investigation.[7] Adler[8] estimated the corrections arising from continuation in q^2 alone in the context of a Born-approximation model. His results are consistent with the PCAC assumption of smooth extrapolation in q^2. Consequently, we assume that continuation of all dynamical parameters in q^2 alone is smooth and consider corrections arising from continuation in s from m_N^2 to $(m_N + m_\pi)^2$ by requiring the off-mass shell amplitude to satisfy the requirements of analyticity and unitarity. In addition, we consider kinematic corrections arising from continuation in q^2.

Specifically, we assume that the off-shell amplitude for s-wave pion-nucleon scattering in the $I=\frac{1}{2}$ channel, $f_{0+}^{1/2}(s,q^2)$, is, for each fixed value of q^2, an analytic function of s in the complex s plane, with cuts analogous to those imposed on the physical amplitude by unitarity and crossing. Thus

$$f_{0+}^{1/2}(s,q^2) = N(s,q^2/D(s,q^2),$$

where $N(s,q^2)$ has the "force" cuts arising from cross-channel exchanges and $D(s,q^2)$ has the unitarity cut. We approximate the cuts in N by a single pole on the negative real s axis[9] and assume a once-subtracted dispersion relation for D, normalizing the amplitude at the subtraction point:

$$f_{0+}^{1/2}(s,q^2) = \frac{R(q^2)}{s+m_0^2}\left[1+\frac{(s-s_0)}{\pi}\int_{s_{\rm th}}^{\infty}\frac{{\rm Im}D(s',q^2)ds'}{(s'-s)(s'-s_0)}\right]^{-1}.$$

$s_{\rm th}$ is the location of the branch point on the positive real s axis at which the unitarity cut begins. We assume that the pion in the lowest mass (i.e., pion-nucleon) intermediate state in the direct channel is a physical pion of mass m_π: Only the external pions are taken off the mass shell (see Adler[8] and Hamilton[10]). Hence, the inelastic thresholds do not collapse into the elastic one and we can use elastic unitarity to write

$${\rm Im}D(s,q^2) = -|\vec{q}|\rho(s)N(s,q^2),$$

where $\rho(s)$ is a phase-space factor[11] and $|\vec{q}|$ is the c.m. momentum in the s channel, given in terms of s and q^2 by off-shell kinematics.[12] Our assumptions about the pion in the intermediate state give $s_{\rm th} = (m_N + m_\pi)^2$, so that

$$f_{0+}^{1/2}(s,q^2) = \frac{R(q^2)}{s+m_0^2}\left[1-\frac{(s-s_0)}{\pi}\int_{(m_N+m_\pi)^2}^{\infty}\frac{ds'|\vec{q}'|\rho(s')R(q^2)}{(s'-s)(s'-s_0)(s'+m_0^2)}\right]^{-1}.$$

We now use current algebra in the soft-pion limit to evaluate $R(0)$. We note that in this limit, the amplitude as given by current algebra and evaluated at $s=m_N^2$ is analytic (in the s wave).[13] Since we are unitarizing this amplitude, our "input" is analytic at this point, which we choose as our subtraction point s_0. Consequently, if a_c is the s wave $I=\frac{1}{2}$ scattering length in the soft-pion limit,

$$a_c \equiv [1/\rho(s)\,{\rm Re}(f_{0+}^{1/2})^{-1}(s,q^2)]_{\substack{s=m_N^2\\ q^2=0}} = R(0)/\rho(m_N^2)(m_0^2+m_N^2) = 0.20 m_\pi^{-1},$$

and we now have

$$f_{0+}^{1/2}(s,q^2) = \frac{R(0)}{s+m_0^2}\left[1-\frac{(s-m_N^2)}{\pi}\int_{(m_N+m_\pi)^2}^{\infty}\frac{|\vec{q}'|\rho(s')R(0)ds'}{(s'-s)(s'-m_N^2)(s'+m_0^2)}\right]^{-1}.$$

The q^2 dependence of the left-hand side is of kinematic origin; this is in contradistinction to the dynamical dependence in $R(q^2)$.

As a consequence of our earlier assumptions,

$$R(0) \approx R(m_\pi^2) \equiv R.$$

Thus $f_{0+}^{1/2}(s,q^2) = R/(s+m_0^2)[1-RI(s,q^2)]$, where

$$I(s,q^2) \equiv \int_{(m_N+m_\pi)^2}^{\infty}\frac{|\vec{q}'|\rho(s')ds'}{(s'-s)(s'-m_N^2)(s'+m_0^2)}\,\frac{(s-m_N^2)}{\pi}$$

is the unitarity correction proportional to m_π at the physical threshold $s=(m_N+m_\pi)^2$. The physical scattering length is

$$a=a_c\frac{\rho(m_N^2)(m_0^2+m_N^2)}{\rho[(m_N+m_\pi)^2][m_0^2+(m_N+m_\pi)^2)]}[1-a_c\rho(m_N^2)(m_N^2+m_0^2)\,\mathrm{Re}I((m_N+m_\pi)^2,m_\pi^2)]^{-1}.$$

The first term in this sum represents the change in the scattering length arising from the change in threshold through kinematics alone; the second term represents the combined corrections arising from unitarity and off-shell kinematics.

As pointed out by Schnitzer,[6] off-shell kinematics plays an important role in the evaluation of the s-wave effective range, since

$$\frac{\partial f(s,q^2)}{\partial \vec{q}^2}=\frac{\partial f}{\partial s}\frac{\partial s}{\partial \vec{q}^2}+\frac{\partial f}{\partial q^2}\frac{\partial q^2}{\partial \vec{q}^2}.$$

Our values are[14,15]

$$a=0.16m_\pi^{-1},$$

$$\tfrac{1}{2}r_0=1.46m_\pi^{-1}.$$

In summary, we have unitarized the soft-pion current-algebra amplitude for s-wave pion-nucleon scattering in the $I=\frac{1}{2}$ channel and moved the threshold up to $s=(m_N+m_\pi)^2$; kinematic corrections, but not the dynamical ones, arising from the zero mass of the external pions have been incorporated. The latter will be the subject of a future investigation.

Calculations are being carried out on similar lines for the $\bar{K}N$ system, where we hope to obtain the imaginary parts of the scattering lengths by coupling unitarity with current algebra.

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[1]M. Gell-Mann, Phys. Rev. 125, 1067 (1962), and Physics 1, 63 (1964).

[2]M. Gell-Mann and M. Levy, Nuovo Cimento 16, 705 (1960); Y. Nambu, Phys. Rev. Letters 4, 380 (1960).

[3]S. Weinberg, Phys. Rev. Letters 17, 616 (1966).

[4]Y. Tomozawa, Nuovo Cimento 46A, 707 (1966).

[5]See, e.g., K. Raman and E. C. Sudarshan, Phys. Rev. 154, 1479 (1967).

[6]H. J. Schnitzer, Phys. Rev. 158, 1471 (1967).

[7]See, e.g., K. Raman, Phys. Rev. Letters 17, 983 (1966); A. P. Balachandran et al., Nuovo Cimento 44A, 1257 (1966); K. Raman, Phys. Rev. 164, 1736 (1967); K. Kang and T. Akiba, Phys. Rev. 164, 1836 (1967).

[8]S. Adler, Phys. Rev. 140, B736 (1965).

[9]Such an approximation necessarily renders our results model dependent; but we are here interested in estimating the magnitude of the unitarity corrections. A more accurate calculation may include physical poles and cuts in N. See S. K. Bose and S. N. Biswas, Phys. Rev. 134, B635 (1964).

[10]J. Hamilton, Nucl. Phys. B1, 449 (1967).

[11]$\rho(s)$ is a phase-space factor; we choose $\rho(s)=\sqrt{s}$. See F. Ferrari, M. Nauenberg, and M. Pusterla, University of California Radiation Laboratory Report No. UCRL 8985, 1959 (unpublished). We work in the s plane: See V. Singh and B. M. Udgaonkar, Phys. Rev. 128, 1820 (1962).

[12]See Schnitzer, Ref. 6.

[13]This assumption is more transparent physically for the s-wave amplitude in the $I=\frac{3}{2}$ channel. However, $a^{3/2}$ and $a^{1/2}$ are related by a simple sum rule in current algebra and we may make an equivalent assumption for our amplitude in our calculations.

[14]We define the low-energy parameters by $\mathrm{Re}(f_{0+}^{1/2})_{\mathrm{th}}^{-1}=1/a+\frac{1}{2}r_0\vec{q}^2\equiv|\vec{q}|\cot\delta$. J. Hamilton and W. S. Woolcock [Rev. Mod. Phys. 35, 737 (1963)] use $\mathrm{Re}(f_{0+}^{1/2})_{\mathrm{th}}=a+\frac{1}{2}r_0'\vec{q}^2\equiv\sin 2\delta/2|\vec{q}|$. Their experimental numbers $a=0.171m_\pi^{-1}$ and $\frac{1}{2}r_0'=-0.022m_\pi^{-3}$ give, with our definitions, $a=0.171m_\pi^{-1}$ and $\frac{1}{2}r_0=0.758m_\pi^{-1}$. Thus our value of the effective range is correct in sign but too large.

[15]The pole position m_0 is an arbitrary parameter. The results quoted are for $m_0=m_N$. We have varied m_0 in the range $4m_N\geqslant m_0\geqslant m_N$ and find that the results do not change appreciably. Larger values of m_0 lead to larger values of a and lower values of r_0.